$\mathscr{C}^m$-SMOOTHNESS OF INVARIANT FIBER BUNDLES FOR DYNAMIC EQUATIONS ON MEASURE CHAINS

CHRISTIAN PÖTZSCHE AND STEFAN SIEGMUND

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We present a new self-contained and rigorous proof of the smoothness of invariant fiber bundles for dynamic equations on measure chains or time scales. Here, an invariant fiber bundle is the generalization of an invariant manifold to the nonautonomous case. Our main result generalizes the "Hadamard-Perron theorem" to the time-dependent, infinite-dimensional, noninvertible, and parameter-dependent case, where the linear part is not necessarily hyperbolic with variable growth rates. As a key feature, our proof works without using complicated technical tools.

1. Introduction

The method of invariant manifolds was originally developed by Lyapunov, Hadamard, and Perron for time-independent diffeomorphisms and ordinary differential equations at a hyperbolic fixed point. It was then extended from hyperbolic to nonhyperbolic systems, from time-independent and finite-dimensional to time-dependent and infinite-dimensional equations, and turned out to be one of the main tools in the contemporary theory of dynamical systems. It is our objective to unify the difference and ordinary differential equations case, and extend them to dynamic equations on measure chains or time scales (closed subsets of the real line). Such equations additionally allow to describe, for example, a hybrid behavior with discrete and continuous dynamical features, or allow an elegant formulation of analytical discretization theory if variable step sizes are present.

This paper can be seen as an immediate continuation of [18], where the existence and $\mathscr{C}^1$-smoothness of invariant fiber bundles for a general class of nonautonomous, noninvertible, and pseudohyperbolic dynamic equations on measure chains have been proved; moreover we obtained a higher-order smoothness for invariant fiber bundles of stable and unstable types therein. While the existence and $\mathscr{C}^1$-smoothness result in [18] is a special case of our main theorem (Theorem 3.5), we additionally prove the differentiability of the fiber bundles under a sharp gap condition using a direct strategy (cf. Theorem 4.2). The differentiability of invariant fiber bundles plays a substantial role in their calculation

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using a Taylor series approach, as well as, for example, in the smooth decoupling of dynamical systems (cf. [5]). To keep the current paper as short as possible, we reduce its contents to a quite technical level. Nonetheless, a variety of applications, examples, outlooks, and further references can be found, for example, in [1, 2, 3, 12].

While in the hyperbolic case the smoothness of the invariant fiber bundles is easily obtained with the uniform contraction principle, in the nonhyperbolic situation the smoothness depends on a spectral gap condition and is subtle to prove. For a modern approach using sophisticated fixed point theorems, see [9, 22, 25, 26]. Another approach to the smoothness of invariant manifolds is essentially based on a lemma by Henry (cf., e.g., [6, Lemma 2.1]) or methods of a more differential topological nature (cf. [11, 23]), namely the $\mathscr{C}^m$-section theorem for fiber-contracting maps. In [5, 20, 24] the problem of higher-order smoothness is tackled directly.

In this spirit we present an accessible "ad hoc" approach to $\mathscr{C}^m$-smoothness of pseudohyperbolic invariant fiber bundles, which is basically derived from [24] (see also [20]) and needs no technical tools beyond the contraction mapping principle, the Neumann series, and Lebesgue's dominated convergence theorem, consequently. Our focus is to give an explicit proof of the higher-order smoothness without sketched induction arguments, but even in the $\mathscr{C}^1$-case, the arguments in this paper are different from those in [18]. One difficulty of the smoothness proof is due to the fact that one has to compute the higher-order derivatives of compositions of maps, the so-called "derivative tree." It turned out to be advantageous to use two different representations of the derivative tree, namely, a "totally unfolded derivative tree" to show that a fixed point operator is well defined and to compute explicit global bounds for the higher-order derivatives of the fiber bundles, and a "partially unfolded derivative tree" to elaborate the induction argument in a recursive way.

Some contemporary results on the higher-order smoothness of invariant manifolds for differential equations can be found, for example, in [6, 22, 24, 25, 26], while corresponding theorems on difference equations are contained in [7, 12]. The first paper [7] deals only with autonomous systems (maps) and applies the fiber contraction theorem. In [12, Theorem 6.2.8, pages 242-243], the so-called Hadamard-Perron theorem is proved via a graph transformation technique for a time-dependent family of $\mathscr{C}^m$-diffeomorphisms on a finite-dimensional space, where higher-order differentiability is only tackled in a hyperbolic situation. Using a different method of proof, our main results, Theorems 3.5 and 4.2, generalize the Hadamard-Perron theorem to noninvertible, infinite-dimensional, and parameter-dependent dynamic equations on measure chains. This enables one to apply our results, for example, in the discretization theory of 2-parameter semiflows. So far, besides [18], there are only three other contributions to the theory of invariant manifolds for dynamic equations on measure chains or time scales. A rigorous proof of the smoothness of generalized center manifolds for autonomous dynamic equations on homogeneous time scales is presented in [9], while [10, Theorem 4.1] shows the existence of a "center fiber bundle" (in our terminology) for nonautonomous systems on measure chains. Finally the thesis [13] deals with classical stable, unstable, and center invariant fiber bundles and their smoothness for dynamic equations on arbitrary time scales, and contains applications to analytical discretization theory.

The structure of the present paper is as follows. In Section 2, we will briefly repeat or collect the notation and basic concepts. In particular, we introduce the elementary calculus on measure chains, dynamic equations, and a convenient notion describing exponential growth of solutions of such equations.

Section 3 will be devoted to the $\mathscr{C}^1$-smoothness of invariant fiber bundles. We will also state our main assumptions here and prove some preparatory lemmas which will also be needed later. The $\mathscr{C}^1$-smoothness follows without any gap condition from the main result of this section, which is Theorem 3.5. Our proof may seem long and intricate and in fact it would be if we would like to show the $\mathscr{C}^1$-smoothness only, but in its structure it already contains the main idea of the induction argument for the $\mathscr{C}^m$-case and we will profit then from being rather detailed in the $\mathscr{C}^1$-case.

Section 4, finally, contains our main result (Theorem 4.2), stating that under the "gap condition" $m_s \odot a \lhd b$ the pseudostable fiber bundle is of class $\mathscr{C}^{m_s}$ and, accordingly, the pseudo-unstable fiber bundle is of class $\mathscr{C}^{m_r}$, if $a \lhd m_r \odot b$.

2. Preliminaries

Above all, to keep the present paper self-contained we repeat some notation from [18]: $\mathbb{N}$ denotes the positive integers. The Banach spaces $\mathscr{X}$, $\mathscr{Y}$ are all real or complex throughout this paper and their norms are denoted by $\|\cdot\|_{\mathscr{X}}$, $\|\cdot\|_{\mathscr{Y}}$, respectively, or simply by $\|\cdot\|$. If $\mathscr{X}$ and $\mathscr{Y}$ are isometrically isomorphic, we write $\mathscr{X} \cong \mathscr{Y}$. $\mathscr{L}_n(\mathscr{X};\mathscr{Y})$ is the Banach space of n-linear continuous operators from $\mathscr{X}^n$ to $\mathscr{Y}$ for $n \in \mathbb{N}$, $\mathscr{L}_0(\mathscr{X};\mathscr{Y}) := \mathscr{Y}$, $\mathscr{L}(\mathscr{X};\mathscr{Y}) := \mathscr{L}_1(\mathscr{X};\mathscr{Y})$, $\mathscr{L}(\mathscr{X}) := \mathscr{L}_1(\mathscr{X};\mathscr{X})$, and $I_{\mathscr{X}}$ stands for the identity map on $\mathscr{X}$. On the product space $\mathscr{X} \times \mathscr{Y}$, we always use the maximum norm

$$\left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|_{\mathscr{X}\times\mathscr{Y}} := \max\{\|x\|_{\mathscr{X}}, \|y\|_{\mathscr{Y}}\}. \tag{2.1}$$

We write DF for the Fréchet derivative of a mapping F, and if $F : (x,y) \mapsto F(x,y)$ depends differentiably on more than one variable, then the partial derivatives are denoted by D_1F and D_2F, respectively. Now we quote the two versions of the higher-order chain rule for Fréchet derivatives on which our smoothness proof is based. Thereto let $\mathscr{Z}$ be a further Banach space over $\mathbb{R}$ or $\mathbb{C}$. With given $j, l \in \mathbb{N}$, we write

$$P_j^<(l) := \left\{ (N_1,\dots,N_j) \;\middle|\; \begin{array}{l} N_i \subseteq \{1,\dots,l\},\, N_i \neq \emptyset \text{ for } i \in \{1,\dots,j\}, \\ N_1 \cup \cdots \cup N_j = \{1,\dots,l\}, \\ N_i \cap N_k = \emptyset \text{ for } i \neq k,\ i,k \in \{1,\dots,j\}, \\ \max N_i < \max N_{i+1} \text{ for } i \in \{1,\dots,j-1\} \end{array} \right\} \tag{2.2}$$

for the set of *ordered partitions* of $\{1,\dots,l\}$ with length j, and $\#N$ for the cardinality of a finite set $N \subset \mathbb{N}$. In case $N = \{n_1,\dots,n_k\} \subseteq \{1,\dots,l\}$ for $k \in \mathbb{N}$, $k \leq l$, we abbreviate $D^k g(x)x_N := D^k g(x)x_{n_1}\cdots x_{n_k}$ for vectors $x, x_1,\dots,x_l \in \mathscr{X}$, where $g : \mathscr{X} \to \mathscr{Y}$ is assumed to be l-times continuously differentiable.

Theorem 2.1 (chain rule). *Given $m \in \mathbb{N}$ and two mappings $f : \mathfrak{Y} \to \mathfrak{Z}$, $g : \mathfrak{X} \to \mathfrak{Y}$ which are m-times continuously differentiable, then also the composition $f \circ g : \mathfrak{X} \to \mathfrak{Z}$ is m-times continuously differentiable and for $l \in \{1,\dots,m\}$, $x \in \mathfrak{X}$, the derivatives possess the representations as a so-called* partially unfolded derivative tree

$$D^l(f \circ g)(x) = \sum_{j=0}^{l-1} \binom{l-1}{j} D^j[Df(g(x))] \cdot D^{l-j}g(x) \tag{2.3}$$

and as a so-called totally unfolded derivative tree

$$D^l(f \circ g)(x)x_1 \cdots x_l = \sum_{j=1}^{l} \sum_{(N_1,\dots,N_j) \in P_j^<(l)} D^j f(g(x)) D^{\#N_1}g(x)x_{N_1} \cdots D^{\#N_j}g(x)x_{N_j} \tag{2.4}$$

for any $x_1,\dots,x_l \in \mathfrak{X}$.

Proof. A proof of (2.3) follows by an easy induction argument (cf. [24, B.3 Satz, page 266]), while (2.4) is shown in [21, Theorem 2]. □

We also introduce some notions which are specific to the calculus on measure chains (cf. [4, 8]). In all the subsequent considerations, we deal with a *measure chain* $(\mathbb{T}, \preceq, \mu)$ unbounded above, that is, a conditionally complete totally ordered set $(\mathbb{T}, \preceq)$ (see [8, Axiom 2]) with the growth calibration $\mu : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ (see [8, Axiom 3]), such that the set $\mu(\mathbb{T}, \tau) \subseteq \mathbb{R}$, $\tau \in \mathbb{T}$, is unbounded above. In addition, $\sigma : \mathbb{T} \to \mathbb{T}$, $\sigma(t) := \inf\{s \in \mathbb{T} : t \prec s\}$, defines the *forward jump operator* and the *graininess* $\mu^* : \mathbb{T} \to \mathbb{R}$, $\mu^*(t) := \mu(\sigma(t), t)$, is assumed to be bounded from now on. A measure chain is called *homogeneous* if its graininess is constant and a *time scale* is the special case of a measure chain, where $\mathbb{T}$ is a canonically ordered closed subset of the reals. For $\tau, t \in \mathbb{T}$, we define

$$(\tau, t)_{\mathbb{T}} := \{s \in \mathbb{T} : \tau \prec s \prec t\}, \qquad \mathbb{T}_\tau^+ := \{s \in \mathbb{T} : \tau \preceq s\}, \qquad \mathbb{T}_\tau^- := \{s \in \mathbb{T} : s \preceq \tau\}, \tag{2.5}$$

and for $N \subseteq \mathbb{T}$, set $N^\kappa := \{t \in \mathbb{N} : t \text{ is not a left-scattered maximum of } N\}$. Following [8, Section 4.1], $\mathscr{C}_{\mathrm{rd}}(\mathbb{T}, \mathscr{L}(\mathfrak{X}))$ and $\mathscr{C}_{\mathrm{rd}}\mathscr{R}(\mathbb{T}, \mathscr{L}(\mathfrak{X}))$ and denote the rd-continuous the rd-continuous regressive functions from $\mathbb{T}$ to $\mathscr{L}(\mathfrak{X})$ (cf. [8, Section 6.1]). Recall that $\mathscr{C}_{\mathrm{rd}}^+\mathscr{R}(\mathbb{T}, \mathbb{R}) := \{c \in \mathscr{C}_{\mathrm{rd}}\mathscr{R}(\mathbb{T}, \mathbb{R}) : 1 + \mu^*(t)a(t) > 0 \text{ for } t \in \mathbb{T}\}$ forms the so-called *regressive module* with respect to the algebraic operations

$$(a \oplus b)(t) := a(t) + b(t) + \mu^*(t)a(t)b(t), \qquad (n \odot a)(t) := \lim_{h \searrow \mu^*(t)} \frac{(1 + ha(t))^n - 1}{h} \tag{2.6}$$

for $t \in \mathbb{T}$, integers n, and $a, b \in \mathscr{C}_{\mathrm{rd}}^+\mathscr{R}(\mathbb{T}, \mathbb{R})$; then a has the additive inverse $(\ominus a)(t) := -a(t)/(1 + \mu^*(t)a(t))$, $t \in \mathbb{T}$. *Growth rates* are functions $a \in \mathscr{C}_{\mathrm{rd}}^+\mathscr{R}(\mathbb{T}, \mathbb{R})$ such that $1 + \inf_{t \in \mathbb{T}} \mu^*(t)a(t) > 0$ and $\sup_{t \in \mathbb{T}} \mu^*(t)a(t) < \infty$ hold. Moreover, we define the relations

$$a \lhd b :\Longleftrightarrow 0 < \lfloor b - a \rfloor := \inf_{t \in \mathbb{T}} (b(t) - a(t)), \qquad a \unlhd b :\Longleftrightarrow 0 \le \lfloor b - a \rfloor, \tag{2.7}$$

and $e_a(t,\tau) \in \mathbb{R}$, $t,\tau \in \mathbb{T}$, stands for the real exponential function on $\mathbb{T}$. Many properties of $e_a(t,\tau)$ used in this paper can be found in [8, Section 7].

Definition 2.2. For a function $c \in \mathscr{C}^+_{\mathrm{rd}}\mathscr{R}(\mathbb{T},\mathbb{R})$, $\tau \in \mathbb{T}$, and an rd-continuous function $\phi : \mathbb{T} \to \mathscr{X}$,

(a) ϕ is *c^+-quasibounded*, if $\|\phi\|^+_{\tau,c} := \sup_{\tau \preceq t} \|\phi(t)\| e_c(\tau,t) < \infty$,
(b) ϕ is *c^--quasibounded*, if $\|\phi\|^-_{\tau,c} := \sup_{t \preceq \tau} \|\phi(t)\| e_c(\tau,t) < \infty$,
(c) ϕ is *$c^\pm$-quasibounded*, if $\sup_{t\in\mathbb{T}} \|\phi(t)\| e_c(\tau,t) < \infty$.

$\mathscr{B}^+_{\tau,c}(\mathscr{X})$ and $\mathscr{B}^-_{\tau,c}(\mathscr{X})$ denote the sets of all c^+- and c^--quasibounded functions $\phi : \mathbb{T} \to \mathscr{X}$, respectively, and they are nontrivial Banach spaces with the norms $\|\cdot\|^+_{\tau,c}$ and $\|\cdot\|^-_{\tau,c}$, respectively.

Lemma 2.3. *For functions $c,d \in \mathscr{C}^+_{rd}\mathscr{R}(\mathbb{T},\mathbb{R})$ with $c \trianglelefteq d$, $m \in \mathbb{N}$, and $\tau \in \mathbb{T}$, the following are true:*

(a) *the Banach spaces $\mathscr{B}^+_{\tau,c}(\mathscr{X}) \times \mathscr{B}^+_{\tau,c}(\mathscr{Y})$ and $\mathscr{B}^+_{\tau,c}(\mathscr{X}\times\mathscr{Y})$ are isometrically isomorphic,*
(b) *$\mathscr{B}^+_{\tau,c}(\mathscr{X}) \subseteq \mathscr{B}^+_{\tau,d}(\mathscr{X})$ and $\|\phi\|^+_{\tau,d} \le \|\phi\|^+_{\tau,c}$ for $\phi \in \mathscr{B}^+_{\tau,c}(\mathscr{X})$,*
(c) *with the abbreviations $\mathscr{B}^0_{\tau,c} := \mathscr{B}^+_{\tau,c}(\mathscr{X}\times\mathscr{Y})$, $\mathscr{B}^m_{\tau,c} := \mathscr{B}^+_{\tau,c}(\mathscr{L}_m(\mathscr{X};\mathscr{X}\times\mathscr{Y}))$, the Banach spaces $\mathscr{B}^m_{\tau,c}$ and $\mathscr{L}(\mathscr{X};\mathscr{B}^{m-1}_{\tau,c})$ are isometrically isomorphic.*

Proof. We only show assertion (c) and refer to [17, Lemma 1.4.6, page 77] for (a) and (b). For that purpose, consider the mapping $J : \mathscr{B}^m_{\tau,c} \to \mathscr{L}(\mathscr{X};\mathscr{B}^{m-1}_{\tau,c})$, $((J\Phi)x)(t) := \Phi(t)x$, for $t \in \mathbb{T}^+_\tau$, $x \in \mathscr{X}$. To prove that J is the wanted norm isomorphism, we choose $\Phi \in \mathscr{B}^m_{\tau,c}$ and a vector $x \in \mathscr{X}$ arbitrarily, and obtain

$$\|\Phi(t)x\|_{\mathscr{L}_{m-1}(\mathscr{X};\mathscr{X}\times\mathscr{Y})} e_c(\tau,t) \le \|\Phi(t)\| e_c(\tau,t)_{\mathscr{L}_m(\mathscr{X};\mathscr{X}\times\mathscr{Y})} \|x\| \le \|\Phi\|^+_{\tau,c}\|x\| \quad \text{for } t \in \mathbb{T}^+_\tau. \tag{2.8}$$

Thus the continuity of the evidently linear map J follows from

$$\|J\Phi\|_{\mathscr{L}(\mathscr{X};\mathscr{B}^{m-1}_{\tau,c})} = \sup_{\|x\|=1} \|(J\Phi)x\|^+_{\tau,c} \le \|\Phi\|^+_{\tau,c}. \tag{2.9}$$

Vice versa, the inverse $J^{-1} : \mathscr{L}(\mathscr{X};\mathscr{B}^{m-1}_{\tau,c}) \to \mathscr{B}^m_{\tau,c}$ of J is given by $(J^{-1}\bar\Phi)(t)x := (\bar\Phi x)(t)$ for $t \in \mathbb{T}^+_\tau$ and $x \in \mathscr{X}$. By the open mapping theorem (cf., e.g., [14, Corollary 1.4, page 388]) J^{-1} is continuous and it remains to show that it is nonexpanding. Thereto we choose $\bar\Phi \in \mathscr{L}(\mathscr{X};\mathscr{B}^{m-1}_{\tau,c})$, $x \in \mathscr{X}$ arbitrarily to get

$$\begin{aligned} \|(J^{-1}\bar\Phi)(t)x\|_{\mathscr{L}_{m-1}(\mathscr{X};\mathscr{X}\times\mathscr{Y})} e_c(\tau,t) &= \|(\bar\Phi x)(t)\|_{\mathscr{L}_{m-1}(\mathscr{X};\mathscr{X}\times\mathscr{Y})} e_c(\tau,t) \\ &\le \|\bar\Phi x\|^+_{\tau,c} \le \|\bar\Phi\|_{\mathscr{L}(\mathscr{X};\mathscr{B}^{m-1}_{\tau,c})}\|x\| \end{aligned} \tag{2.10}$$

for $t \in \mathbb{T}^+_\tau$, and this estimate yields $\|(J^{-1}\bar\Phi)(t)\|_{\mathscr{L}_m(\mathscr{X};\mathscr{X}\times\mathscr{Y})} e_c(\tau,t) \le \|\bar\Phi\|_{\mathscr{L}(\mathscr{X};\mathscr{B}^{m-1}_{\tau,c})}$, which in turn ultimately gives us the desired $\|J^{-1}\bar\Phi\|^+_{\tau,c} \le \|\bar\Phi\|_{\mathscr{L}(\mathscr{X};\mathscr{B}^{m-1}_{\tau,c})}$. Consequently, J is an isometry. $\square$

A mapping $\phi : \mathbb{T} \to \mathscr{X}$ is said to be *differentiable* (at some $t_0 \in \mathbb{T}$) if there exists a unique *derivative* $\phi^\Delta(t_0) \in \mathscr{X}$ such that for any $\epsilon > 0$, the estimate

$$\|\phi(\sigma(t_0)) - \phi(t) - \mu(\sigma(t_0),t)\phi^\Delta(t_0)\| \le \epsilon \,|\mu(\sigma(t_0),t)| \quad \text{for } t \in U, \tag{2.11}$$

holds in a $\mathbb{T}$-neighborhood U of t_0 (see [8, Section 2.4]). We write $\Delta_1 s : \mathbb{T} \times \mathscr{X} \to \mathscr{Y}$ for the partial derivative with respect to the first variable of a mapping $s : \mathbb{T} \times \mathscr{X} \to \mathscr{Y}$, provided it exists. The (Lebesgue) integral of $\phi : \mathbb{T} \to \mathscr{X}$ is denoted by $\int_\tau^t \phi(s)\Delta s$, provided again it exists (cf. [16]).

Now let $\mathscr{P}$ be a nonempty set, momentarily. For a dynamic equation

$$x^\Delta = f(t,x,p) \tag{2.12}$$

with a right-hand side $f : \mathbb{T} \times \mathscr{X} \times \mathscr{P} \to \mathscr{X}$ guaranteeing existence and uniqueness of solutions in forward time (see, e.g., [17, Satz 1.2.17(a), page 38]), let $\varphi(t;\tau,\xi,p)$ denote the *general solution*, that is, $\varphi(\cdot;\tau,\xi,p)$ solves (2.12) on $\mathbb{T}_\tau^+ \cap I$, I is a $\mathbb{T}$-interval, and satisfies the initial condition $\varphi(\tau;\tau,\xi,p) = \xi$ for $\tau \in I$, $\xi \in \mathscr{X}$, and $p \in \mathscr{P}$. As mentioned in the introduction, invariant fiber bundles are generalizations of invariant manifolds to nonautonomous equations. In order to be more precise, for fixed parameters $p \in \mathscr{P}$, we call a subset $S(p)$ of the extended state space $\mathbb{T} \times \mathscr{X}$ an *invariant fiber bundle* of (2.12) if it is *positively invariant*, that is, for any pair $(\tau,\xi) \in S(p)$, one has $(t,\varphi(t;\tau,\xi,p)) \in S(p)$ for all $t \in \mathbb{T}_\tau^+$. At this point it is appropriate to state an existence and uniqueness theorem for (2.12) which is sufficient for our purposes.

Theorem 2.4. *Assume that $f : \mathbb{T} \times \mathscr{X} \times \mathscr{P} \to \mathscr{X}$ satisfies the following conditions:*
(i) *$f(\cdot,p)$ is rd-continuous for every $p \in \mathscr{P}$,*
(ii) *for each $t \in \mathbb{T}$, there exist a compact $\mathbb{T}$-neighborhood N_t and a real $l_0(t) \ge 0$ such that*

$$\|f(s,x,p) - f(s,\bar{x},p)\| \le l_0(t)\|x - \bar{x}\| \quad \text{for } s \in N_t^\kappa,\ x,\bar{x} \in \mathscr{X},\ p \in \mathscr{P}. \tag{2.13}$$

Then the following hold:
(a) *for each $\tau \in \mathbb{T}$, $\xi \in \mathscr{X}$, $p \in \mathscr{P}$, the solution $\varphi(\cdot;\tau,\xi,p)$ is uniquely determined and exists on a $\mathbb{T}$-interval I such that $\mathbb{T}_\tau^+ \subseteq I$ and I is a $\mathbb{T}$-neighborhood of τ independent of $\xi \in \mathscr{X}$, $p \in \mathscr{P}$;*
(b) *if $\xi : \mathscr{P} \to \mathscr{X}$ is bounded and if there exists an rd-continuous mapping $l_1 : \mathbb{T} \to \mathbb{R}_0^+$ such that*

$$\|f(t,x,p)\| \le l_1(t)\|x\| \quad \text{for } (t,x,p) \in \mathbb{T} \times \mathscr{X} \times \mathscr{P}, \tag{2.14}$$

then $\lim_{t\to\tau} \varphi(t;\tau,\xi(p),p) = \xi(p)$ holds uniformly in $p \in \mathscr{P}$.

Proof. (a) The existence and uniqueness of $\varphi(\cdot;\tau,\xi,p)$ on $\mathbb{T}_\tau^+$ are basically shown in [8, Theorem 5.7] (cf. also [17, Satz 1.2.17(a), page 38]). In a left-scattered $\tau \in \mathbb{T}$, we choose $I := \mathbb{T}_\tau^+$, while in a left-dense point $\tau \in \mathbb{T}$, the solution $\varphi(\cdot;\tau,\xi,p)$ exists in a whole $\mathbb{T}$-neighborhood of τ due to [8, Theorem 5.5]. This neighborhood does not depend on $\xi \in \mathscr{X}$, $p \in \mathscr{P}$ since (2.13) holds uniformly in $x \in \mathscr{X}$, $p \in \mathscr{P}$.

(b) Let N be a compact $\mathbb{T}$-neighborhood of τ such that $\varphi(\cdot;\tau,\xi(p),p)$ exists on $N \cup \mathbb{T}_\tau^+$. Then the estimate

$$\begin{aligned}||\varphi(t;\tau,\xi(p),p)|| &\leq ||\xi(p)|| + \int_\tau^t ||f(s,\varphi(s;\tau,\xi(p),p),p)||\Delta s \\ &\leq \sup_{p\in\mathcal{P}} ||\xi(p)|| + \int_\tau^t l_1(s)||\varphi(s,\tau,\xi(p),p)||\Delta s \quad \text{by (2.14)},\end{aligned} \tag{2.15}$$

for $t \in \mathbb{T}_\tau^+$, is valid, and with Gronwall's lemma (cf., e.g., [17, Korollar 1.3.31, page 66]), we get

$$||\varphi(t;\tau,\xi(p),p)|| \leq \sup_{p\in\mathcal{P}} ||\xi(p)|| e_{l_1}(t,\tau) \quad \text{for } t \in \mathbb{T}_\tau^+. \tag{2.16}$$

On the other hand, if $\tau \in \mathbb{T}$ is left-dense, we obtain $\lim_{t\nearrow\tau}\mu^*(t) = 0$ and consequently $l_1(t)\mu^*(t) < 1$ holds for $t \prec \tau$ in a $\mathbb{T}$-neighborhood, without loss of generality, N of τ. Then $-l_1$ is positively regressive, and similar to (2.16), we obtain $\|\varphi(t;\tau,\xi(p),p)\| \leq \sup_{p\in\mathcal{P}} \|\xi(p)\| e_{-l_1}(t,\tau)$ for $t \prec \tau$, $t \in \mathbb{N}$. Hence, because of the compactness of N and the continuity of $e_{l_1}(\cdot,\tau)$, $e_{-l_1}(\cdot,\tau)$, there exists a $C \geq 0$ with $\|\varphi(t;\tau,\xi(p),p)\| \leq C$ for all $t \in \mathbb{N}$, $p \in \mathcal{P}$, and this implies

$$\begin{aligned}||\varphi(t;\tau,\xi(p),p) - \xi(p)|| &\leq \left| \int_\tau^t ||f(s,\varphi(s;\tau,\xi(p),p),p)||\Delta s \right| \\ &\leq \left| \int_\tau^t l_1(s)||\varphi(s;\tau,\xi(p),p)||\Delta s \right| \quad \text{by (2.14)} \\ &\leq C \left| \int_\tau^t l_1(s)\Delta s \right| \xrightarrow[t\to\tau]{} 0\end{aligned} \tag{2.17}$$

uniformly in $p \in \mathcal{P}$, since the right-hand side is independent of p. □

Finally, given $A \in \mathscr{C}_{\mathrm{rd}}(\mathbb{T},\mathscr{L}(\mathscr{X}))$, the *transition operator* $\Phi_A(t,\tau) \in \mathscr{L}(\mathscr{X})$, $\tau \preceq t$, of a linear dynamic equation $x^\Delta = A(t)x$ is the solution of the operator-valued initial value problem $X^\Delta = A(t)X$, $X(\tau) = I_\mathscr{X}$ in $\mathscr{L}(\mathscr{X})$. If A is regressive, then $\Phi_A(t,\tau)$ is defined for all $\tau, t \in \mathbb{T}$.

3. $\mathscr{C}^1$-smoothness of invariant fiber bundles

We begin this section by stating our frequently used main assumptions.

Hypothesis 3.1. Let $\mathcal{P}$ be a locally compact topological space satisfying the first axiom of countability. Consider the system of parameter-dependent dynamic equations

$$x^\Delta = A(t)x + F(t,x,y,p), \qquad y^\Delta = B(t)y + G(t,x,y,p), \tag{3.1}$$

where $A \in \mathscr{C}_{\mathrm{rd}}(\mathbb{T},\mathscr{L}(\mathscr{X}))$, $B \in \mathscr{C}_{\mathrm{rd}}\mathscr{R}(\mathbb{T},\mathscr{L}(\mathscr{Y}))$, and rd-continuous mappings $F : \mathbb{T}\times\mathscr{X}\times\mathscr{Y}\times\mathcal{P} \to \mathscr{X}$, $G : \mathbb{T}\times\mathscr{X}\times\mathscr{Y}\times\mathcal{P} \to \mathscr{Y}$, which are m-times rd-continuously differentiable with respect to (x, y), such that the partial derivatives $D^n_{(2,3)}(F, G)(t, \cdot)$, $t \in \mathbb{T}$, are continuous for $n \in \{0, \ldots, m\}$ and $m \in \mathbb{N}$. Moreover, we assume the following hypotheses.

(i) Hypothesis on linear part. The transition operators $\Phi_A(t,s)$ and $\Phi_B(t,s)$, respectively, satisfy for all $t, s \in \mathbb{T}$ the estimates

$$\begin{aligned} \|\Phi_A(t,s)\|_{\mathscr{L}(\mathscr{X})} &\le K_1 e_a(t,s) \quad \text{for } s \preceq t, \\ \|\Phi_B(t,s)\|_{\mathscr{L}(\mathscr{Y})} &\le K_2 e_b(t,s) \quad \text{for } t \preceq s, \end{aligned} \tag{3.2}$$

with real constants $K_1, K_2 \ge 1$ and growth rates $a, b \in \mathscr{C}^+_{\text{rd}}\mathscr{R}(\mathbb{T}, \mathbb{R})$, $a \lhd b$.

(ii) Hypothesis on perturbation. We have

$$F(t, 0, 0, p) \equiv 0, \quad G(t, 0, 0, p) \equiv 0 \quad \text{on } \mathbb{T} \times \mathscr{P}, \tag{3.3}$$

the partial derivatives of F and G are globally bounded, that is, for each $n \in \{1, \ldots, m\}$, we suppose

$$\begin{aligned} |F|_n &:= \sup_{(t,x,y,p) \in \mathbb{T} \times \mathscr{X} \times \mathscr{Y} \times \mathscr{P}} \|D^n_{(2,3)} F(t, x, y, p)\|_{\mathscr{L}_n(\mathscr{X} \times \mathscr{Y}; \mathscr{X})} < \infty, \\ |G|_n &:= \sup_{(t,x,y,p) \in \mathbb{T} \times \mathscr{X} \times \mathscr{Y} \times \mathscr{P}} \|D^n_{(2,3)} G(t, x, y, p)\|_{\mathscr{L}_n(\mathscr{X} \times \mathscr{Y}; \mathscr{Y})} < \infty, \end{aligned} \tag{3.4}$$

and additionally, for some real $\sigma_{\max} > 0$, we require

$$\max\{|F|_1, |G|_1\} < \frac{\sigma_{\max}}{\max\{K_1, K_2\}}. \tag{3.5}$$

Finally, we choose a fixed real number $\sigma \in (\max\{K_1, K_2\} \max\{|F|_1, |G|_1\}, \sigma_{\max})$.

Remark 3.2. (1) Under Hypothesis 3.1, the above dynamic equation (3.1) satisfies the assumptions of Theorem 2.4 on the Banach space $\mathscr{X} \times \mathscr{Y}$ equipped with the norm (2.1), and therefore its solutions exist and are unique on a $\mathbb{T}$-interval unbounded above.

(2) In [18] we have considered dynamic equations of the type (3.1) without an explicit parameter-dependence and under the assumption that $D^m_{(2,3)}(F, G)$ is uniformly continuous in $t \in \mathbb{T}$. Anyhow, the results from [18] used below remain applicable since all the above estimates in Hypothesis 3.1 are uniform in $p \in \mathscr{P}$ and since the uniform continuity of $D^m_{(2,3)}(F, G)$ is not used to derive them.

Lemma 3.3. *Assume Hypothesis 3.1 for $m = 1$, $\sigma_{\max} = \lfloor b - a \rfloor / 2$, and choose $\tau \in \mathbb{T}$. Moreover, let $(\nu, \upsilon), (\bar{\nu}, \bar{\upsilon}) : \mathbb{T}^+_\tau \to \mathscr{X} \times \mathscr{Y}$ be solutions of (3.1) such that their difference $(\nu, \upsilon) - (\bar{\nu}, \bar{\upsilon})$ is c^+-quasibounded for any $c \in \mathscr{C}^+_{rd}\mathscr{R}(\mathbb{T}, \mathbb{R})$, $a + \sigma \unlhd c \unlhd b - \sigma$. Then the estimate*

$$\left\| \begin{pmatrix} \nu \\ \upsilon \end{pmatrix}(t) - \begin{pmatrix} \bar{\nu} \\ \bar{\upsilon} \end{pmatrix}(t) \right\|_{\mathscr{X} \times \mathscr{Y}} \le K_1 \frac{\lfloor c - a \rfloor}{\lfloor c - a \rfloor - K_1 |F|_1} e_c(t, \tau) \|\nu(\tau) - \bar{\nu}(\tau)\|_{\mathscr{X}} \quad \text{for } t \in \mathbb{T}^+_\tau, \tag{3.6}$$

holds.

Proof. Choose arbitrary $p \in \mathcal{P}$ and $\tau \in \mathbb{T}$. First of all, the difference $\nu - \bar{\nu} \in \mathcal{B}^+_{\tau,c}(\mathcal{X})$ is a solution of the inhomogeneous dynamic equation

$$x^\Delta = A(t)x + F(t,(\nu,\upsilon)(t),p) - F(t,(\bar{\nu},\bar{\upsilon})(t),p), \tag{3.7}$$

where the inhomogeneity is c^+-quasibounded:

$$\|F(\cdot,(\nu,\upsilon)(\cdot),p) - F(\cdot,(\bar{\nu},\bar{\upsilon})(\cdot),p)\|^+_{\tau,c} \leq |F|_1 \left\| \begin{pmatrix}\nu\\ \upsilon\end{pmatrix} - \begin{pmatrix}\bar{\nu}\\ \bar{\upsilon}\end{pmatrix} \right\|^+_{\tau,c} \quad \text{by (3.4)} \tag{3.8}$$

by Hypothesis 3.1(ii). Applying [19, Theorem 2(a)] to (3.7) yields

$$\|\nu - \bar{\nu}\|^+_{\tau,c} \leq K_1 \|\nu(\tau) - \bar{\nu}(\tau)\| + \frac{K_1|F|_1}{\lfloor c-a \rfloor} \left\| \begin{pmatrix}\nu\\ \upsilon\end{pmatrix} - \begin{pmatrix}\bar{\nu}\\ \bar{\upsilon}\end{pmatrix} \right\|^+_{\tau,c}. \tag{3.9}$$

Because of $K_1|F|_1/\lfloor c-a \rfloor < 1$ (cf. (3.5)), without loss of generality, we can assume $\upsilon \neq \bar{\upsilon}$ from now on. Analogously, the difference $\upsilon - \bar{\upsilon} \in \mathcal{B}^+_{\tau,c}(\mathcal{Y})$ is a solution of the linear dynamic equation

$$y^\Delta = B(t)y + G(t,(\nu,\upsilon)(t),p) - G(t,(\bar{\nu},\bar{\upsilon})(t),p), \tag{3.10}$$

where the inhomogeneity is also c^+-quasibounded:

$$\|G(\cdot,(\nu,\upsilon)(\cdot),p) - G(\cdot,(\bar{\nu},\bar{\upsilon})(\cdot),p)\|^+_{\tau,c} \leq |G|_1 \left\| \begin{pmatrix}\nu\\ \upsilon\end{pmatrix} - \begin{pmatrix}\bar{\nu}\\ \bar{\upsilon}\end{pmatrix} \right\|^+_{\tau,c} \quad \text{by (3.4)} \tag{3.11}$$

by Hypothesis 3.1(ii). Now using the result [19, Theorem 4(b)] yields

$$\|\upsilon - \bar{\upsilon}\|^+_{\tau,c} \leq \frac{K_2|G|_1}{\lfloor b-c \rfloor} \left\| \begin{pmatrix}\nu\\ \upsilon\end{pmatrix} - \begin{pmatrix}\bar{\nu}\\ \bar{\upsilon}\end{pmatrix} \right\|^+_{\tau,c}, \tag{3.12}$$

and since we have $K_2|G|_1/\lfloor b-c \rfloor < 1$ (cf. (3.5)), as well as $\upsilon \neq \bar{\upsilon}$, we get the inequality $\|\upsilon - \bar{\upsilon}\|^+_{\tau,c} < \max\{\|\nu - \bar{\nu}\|^+_{\tau,c}, \|\upsilon - \bar{\upsilon}\|^+_{\tau,c}\}$ by (2.1). Consequently, we obtain $\|\nu - \bar{\nu}\|^+_{\tau,c} = \|(\nu,\upsilon) - (\bar{\nu},\bar{\upsilon})\|^+_{\tau,c}$, which leads to

$$\left\| \begin{pmatrix}\nu\\ \upsilon\end{pmatrix} - \begin{pmatrix}\bar{\nu}\\ \bar{\upsilon}\end{pmatrix} \right\|^+_{\tau,c} \leq K_1 \|\nu(\tau) - \bar{\nu}(\tau)\| + \frac{K_1|F|_1}{\lfloor c-a \rfloor} \left\| \begin{pmatrix}\nu\\ \upsilon\end{pmatrix} - \begin{pmatrix}\bar{\nu}\\ \bar{\upsilon}\end{pmatrix} \right\|^+_{\tau,c} \quad \text{by (3.9).} \tag{3.13}$$

This, in turn, immediately implies the estimate (3.6) by Definition 2.2(a). □

Now we collect some crucial results from the earlier paper [18]. In particular, we can characterize the quasibounded solutions of the dynamic equation (3.1) easily as fixed points of an appropriate operator.

Lemma 3.4 (the operator $\mathcal{T}_\tau$). *Assume Hypothesis 3.1 for $m = 1$, $\sigma_{\max} = \lfloor b - a \rfloor/2$, and choose $\tau \in \mathbb{T}$. Then for arbitrary growth rates $c \in \mathscr{C}^+_{rd}\mathscr{R}(\mathbb{T},\mathbb{R})$, $a + \sigma \trianglelefteq c \trianglelefteq b - \sigma$, and $\xi \in \mathscr{X}$, $p \in \mathscr{P}$, the mapping $\mathcal{T}_\tau : \mathscr{B}^+_{\tau,c}(\mathscr{X} \times \mathscr{Y}) \times \mathscr{X} \times \mathscr{P} \to \mathscr{B}^+_{\tau,c}(\mathscr{X} \times \mathscr{Y})$,*

$$\mathcal{T}_\tau(\nu, \upsilon; \xi, p) := \begin{pmatrix} \Phi_A(\cdot,\tau)\xi + \int_\tau^\cdot \Phi_A(\cdot,\sigma(s))F(s,(\nu,\upsilon)(s),p)\Delta s \\ -\int_\cdot^\infty \Phi_B(\cdot,\sigma(s))G(s,(\nu,\upsilon)(s),p)\Delta s \end{pmatrix}, \tag{3.14}$$

has the following properties:

(a) *$\mathcal{T}_\tau(\cdot;\xi,p)$ is a uniform contraction in $\xi \in \mathscr{X}$, $p \in \mathscr{P}$ with Lipschitz constant*

$$L := \frac{\max\{K_1, K_2\}}{\sigma} \max\{|F|_1, |G|_1\} < 1, \tag{3.15}$$

(b) *the unique fixed point $(\nu_\tau, \upsilon_\tau)(\xi, p) \in \mathscr{B}^+_{\tau,c}(\mathscr{X} \times \mathscr{Y})$ of $\mathcal{T}_\tau(\cdot;\xi,p)$ does not depend on $c \in \mathscr{C}^+_{rd}\mathscr{R}(\mathbb{T},\mathbb{R})$, $a + \sigma \trianglelefteq c \trianglelefteq b - \sigma$, and is globally Lipschitzian:*

$$\left\| \begin{pmatrix} \nu_\tau \\ \upsilon_\tau \end{pmatrix}(\xi,p) - \begin{pmatrix} \nu_\tau \\ \upsilon_\tau \end{pmatrix}(\bar{\xi},p) \right\|^+_{\tau,c} \le \frac{K_1}{1-L}\|\xi - \bar{\xi}\|_{\mathscr{X}} \quad \text{for } \xi, \bar{\xi} \in \mathscr{X},\ p \in \mathscr{P}, \tag{3.16}$$

(c) *a function $(\nu, \upsilon) \in \mathscr{B}^+_{\tau,c}(\mathscr{X} \times \mathscr{Y})$ is a solution of the dynamic equation (3.1), with $\nu(\tau) = \xi$, if and only if it is a solution of the fixed point equation*

$$\begin{pmatrix} \nu \\ \upsilon \end{pmatrix} = \mathcal{T}_\tau(\nu, \upsilon; \xi, p). \tag{3.17}$$

Proof. See [18, proof of Theorem 4.9] for assertions (a), (b), and [18, Lemma 4.8] for (c). □

Having all preparatory results at hand, we may now head for our main theorem in the $\mathscr{C}^1$-case.

Theorem 3.5 ($\mathscr{C}^1$-smoothness). *Assume Hypothesis 3.1 for $m = 1$, $\sigma_{\max} = \lfloor b - a \rfloor/2$, and let φ denote the general solution of (3.1). Then the following statements are true.*

(a) *There exists a uniquely determined mapping $s : \mathbb{T} \times \mathscr{X} \times \mathscr{P} \to \mathscr{Y}$ whose graph $S(p) := \{(\tau, \xi, s(\tau,\xi,p)) : \tau \in \mathbb{T},\ \xi \in \mathscr{X}\}$ can be characterized dynamically for any parameter $p \in \mathscr{P}$ and any growth rate $c \in \mathscr{C}^+_{rd}\mathscr{R}(\mathbb{T},\mathbb{R})$, $a + \sigma \trianglelefteq c \trianglelefteq b - \sigma$, as*

$$S(p) = \{(\tau, \xi, \eta) \in \mathbb{T} \times \mathscr{X} \times \mathscr{Y} : \varphi(\cdot;\tau,\xi,\eta,p) \in \mathscr{B}^+_{\tau,c}(\mathscr{X} \times \mathscr{Y})\}. \tag{3.18}$$

Furthermore,

(a_1) *$s(\tau, 0, p) \equiv 0$ on $\mathbb{T} \times \mathscr{P}$,*

(a_2) *$s : \mathbb{T} \times \mathscr{X} \times \mathscr{P} \to \mathscr{Y}$ is continuous, rd-continuously differentiable in the first argument and continuously differentiable in the second argument with globally bounded derivative*

$$\|D_2 s(\tau,\xi,p)\|_{\mathscr{L}(\mathscr{X};\mathscr{Y})} \le \frac{K_1 K_2 \max\{|F|_1, |G|_1\}}{\sigma - \max\{K_1, K_2\} \max\{|F|_1, |G|_1\}} \quad \text{for } (\tau,\xi,p) \in \mathbb{T} \times \mathscr{X} \times \mathscr{P}, \tag{3.19}$$

(a$_3$) *the graph $S(p)$, $p \in \mathcal{P}$, is an invariant fiber bundle of (3.1). Additionally, s is a solution of the* invariance equation

$$\begin{aligned}
&\Delta_1 s(\tau,\xi,p)\\
&\quad = B(\tau)s(\tau,\xi,p) + G(\tau,\xi,s(\tau,\xi,p),p)\\
&\qquad - \int_0^1 D_2 s(\sigma(\tau),\xi + h\mu^*(\tau)[A(\tau)\xi + F(\tau,\xi,s(\tau,\xi,p),p)],p)\,dh\\
&\qquad\quad \times [A(\tau)\xi + F(\tau,\xi,s(\tau,\xi,p),p)]
\end{aligned} \tag{3.20}$$

for $(\tau,\xi,p) \in \mathbb{T}\times\mathcal{X}\times\mathcal{P}$.

The graph $S(p)$, $p \in \mathcal{P}$, is called the pseudostable fiber bundle *of (3.1).*

(b) *In case $\mathbb{T}$ is unbounded below, there exists a uniquely determined mapping $r : \mathbb{T}\times\mathcal{Y}\times\mathcal{P} \to \mathcal{X}$ whose graph $R(p) := \{(\tau, r(\tau,\eta,p),\eta) : \tau \in \mathbb{T},\ \eta \in \mathcal{Y}\}$ can be characterized dynamically for any parameter $p \in \mathcal{P}$ and any growth rate $c \in \mathcal{C}^+_{rd}\mathcal{R}(\mathbb{T},\mathbb{R})$, $a+\sigma \trianglelefteq c \trianglelefteq b-\sigma$, as*

$$R(p) = \{(\tau,\xi,\eta) \in \mathbb{T}\times\mathcal{X}\times\mathcal{Y} : \varphi(\cdot;\tau,\xi,\eta,p) \in \mathcal{B}^-_{\tau,c}(\mathcal{X}\times\mathcal{Y})\}. \tag{3.21}$$

Furthermore

(b$_1$) $r(\tau,0,p) \equiv 0$ *on* $\mathbb{T}\times\mathcal{P}$,

(b$_2$) *$r : \mathbb{T}\times\mathcal{Y}\times\mathcal{P} \to \mathcal{X}$ is continuous, rd-continuously differentiable in the first argument and continuously differentiable in the second argument with globally bounded derivative*

$$\|D_2 r(\tau,\eta,p)\|_{\mathcal{L}(\mathcal{Y};\mathcal{X})} \le \frac{K_1K_2 \max\{|F|_1,|G|_1\}}{\sigma - \max\{K_1,K_2\}\max\{|F|_1,|G|_1\}} \quad \text{for } (\tau,\eta,p) \in \mathbb{T}\times\mathcal{Y}\times\mathcal{P}, \tag{3.22}$$

(b$_3$) *the graph $R(p)$, $p \in \mathcal{P}$, is an invariant fiber bundle of (3.1). Additionally, r is a solution of the* invariance equation

$$\begin{aligned}
&\Delta_1 r(\tau,\eta,p)\\
&\quad = A(\tau)r(\tau,\eta,p) + F(\tau,r(\tau,\eta,p),\eta,p)\\
&\qquad - \int_0^1 D_2 r(\sigma(\tau),\eta + h\mu^*(\tau)[B(\tau)\eta + G(\tau,r(\tau,\eta,p),\eta,p)])\,dh\\
&\qquad\quad \times [B(\tau)\eta + G(\tau,r(\tau,\eta,p),\eta,p)]
\end{aligned} \tag{3.23}$$

for $(\tau,\eta,p) \in \mathbb{T}\times\mathcal{Y}\times\mathcal{P}$.

The graph $R(p)$, $p \in \mathcal{P}$, is called the pseudo-unstable fiber bundle *of (3.1).*

(c) *In case $\mathbb{T}$ is unbounded below, only the zero solution of (3.1) is contained in both $S(p)$ and $R(p)$, that is, $S(p)\cap R(p) = \mathbb{T}\times\{0\}\times\{0\}$ for $p \in \mathcal{P}$, and hence the zero solution is the only $c^{\pm}$-quasibounded solution of (3.1) for $c \in \mathcal{C}^+_{rd}\mathcal{R}(\mathbb{T},\mathbb{R})$, $a+\sigma \trianglelefteq c \trianglelefteq b-\sigma$.*

Remark 3.6. Since we did not assume regressivity of the dynamic equation (3.1), one has to interpret the dynamical characterization (3.21) of the pseudo-unstable fiber bundle

$R(p)$, $p \in \mathcal{P}$, as follows. For fixed $p \in \mathcal{P}$, a point $(\tau,\xi,\eta) \in \mathbb{T}\times\mathcal{X}\times\mathcal{Y}$ is contained in $R(p)$ if and only if there exists a c^--quasibounded solution $\varphi(\cdot;\tau,\xi,\eta,p):\mathbb{T}\to\mathcal{X}\times\mathcal{Y}$ of (3.1) satisfying the initial condition $x(\tau)=\xi$, $y(\tau)=\eta$. In this case the solution $\varphi(\cdot;\tau,\xi,\eta,p)$ is uniquely determined.

Proof. (a) Our main intention in the current proof is to show the continuity and the partial Fréchet differentiability assertion (a_2) for the mapping $s:\mathbb{T}\times\mathcal{X}\times\mathcal{P}\to\mathcal{Y}$. Any other statement from Theorem 3.5(a) follows from [18, proof of Theorem 4.9]. Nevertheless, we reconsider the main ingredients in our argumentation.

Using just [18, proof of Theorem 4.9], we know that for any triple $(\tau,\xi,p)\in\mathbb{T}\times\mathcal{X}\times\mathcal{P}$, there exists exactly one $s(\tau,\xi,p)\in\mathcal{Y}$ such that $\varphi(\cdot;\tau,\xi,s(\tau,\xi,p),p)\in\mathcal{B}^+_{\tau,c}(\mathcal{X}\times\mathcal{Y})$ for every $c\in\mathscr{C}^+_{\mathrm{rd}}\mathcal{R}(\mathbb{T},\mathbb{R})$, $a+\sigma\trianglelefteq c\trianglelefteq b-\sigma$. Then the function $s(\cdot,p):\mathbb{T}\times\mathcal{X}\to\mathcal{Y}$, $p\in\mathcal{P}$, defines the invariant fiber bundle $S(p)$ if we set $s(\tau,\xi,p):=(\upsilon_\tau(\xi,p))(\tau)$, where $(\nu_\tau,\upsilon_\tau)(\xi,p)\in\mathcal{B}^+_{\tau,c}(\mathcal{X}\times\mathcal{Y})$ denotes the unique fixed point of the operator $\mathcal{T}_\tau(\cdot;\xi,p):\mathcal{B}^+_{\tau,c}(\mathcal{X}\times\mathcal{Y})\to\mathcal{B}^+_{\tau,c}(\mathcal{X}\times\mathcal{Y})$ introduced in Lemma 3.4 for any $\xi\in\mathcal{X}$, $p\in\mathcal{P}$, and $c\in\mathscr{C}^+_{\mathrm{rd}}\mathcal{R}(\mathbb{T},\mathbb{R})$, $a+\sigma\trianglelefteq c\trianglelefteq b-\sigma$. Here and in the following, one should be aware of the estimate

$$\max\left\{\frac{K_1|F|_1}{\lfloor c-a\rfloor},\frac{K_2|G|_1}{\lfloor b-c\rfloor}\right\}\le L<1\quad\text{by (3.15)}.\tag{3.24}$$

The further proof of part (a_2) will be subdivided into several steps. For notational convenience, we introduce the abbreviations $\nu_\tau(t;\xi,p):=(\nu_\tau(\xi,p))(t)$ and $\upsilon_\tau(t;\xi,p):=(\upsilon_\tau(\xi,p))(t)$.

Step 1. Claim: *for every growth rate $c\in\mathscr{C}^+_{rd}\mathcal{R}(\mathbb{T},\mathbb{R})$, $a+\sigma\lhd c\trianglelefteq b-\sigma$, the mappings $(\nu_\tau,\upsilon_\tau):\mathcal{X}\times\mathcal{P}\to\mathcal{B}^+_{\tau,c}(\mathcal{X}\times\mathcal{Y})$ and $(\nu_\tau,\upsilon_\tau)(t;\cdot):\mathcal{X}\times\mathcal{P}\to\mathcal{X}\times\mathcal{Y}$, $t\in\mathbb{T}^+_\tau$, are continuous.*

By Hypothesis 3.1, the parameter space $\mathcal{P}$ satisfies the first axiom of countability. Consequently, for example, [15, Theorem 1.1(b), page 190] implies that in order to prove the continuity of the mapping $(\nu_\tau,\upsilon_\tau)(\xi_0,\cdot):\mathcal{P}\to\mathcal{B}^+_{\tau,c}(\mathcal{X}\times\mathcal{Y})$, it suffices to show for arbitrary but fixed $\xi_0\in\mathcal{X}$ and $p_0\in\mathcal{P}$ the following limit relation:

$$\lim_{p\to p_0}\begin{pmatrix}\nu_\tau\\ \upsilon_\tau\end{pmatrix}(\xi_0,p)=\begin{pmatrix}\nu_\tau\\ \upsilon_\tau\end{pmatrix}(\xi_0,p_0)\quad\text{in }\mathcal{B}^+_{\tau,c}(\mathcal{X}\times\mathcal{Y}).\tag{3.25}$$

For any parameter $p\in\mathcal{P}$, we obtain, by using (3.14) and (3.17),

$$\begin{aligned}&\left\|\begin{pmatrix}\nu_\tau\\ \upsilon_\tau\end{pmatrix}(t;\xi_0,p)-\begin{pmatrix}\nu_\tau\\ \upsilon_\tau\end{pmatrix}(t;\xi_0,p_0)\right\|\\
&\le\max\Big\{K_1\int_\tau^t e_a(t,\sigma(s))\|F(s,(\nu_\tau,\upsilon_\tau)(s;\xi_0,p),p)-F(s,(\nu_\tau,\upsilon_\tau)(s;\xi_0,p_0),p_0)\|\Delta s,\\
&\qquad K_2\int_t^\infty e_b(t,\sigma(s))\|G(s,(\nu_\tau,\upsilon_\tau)(s;\xi_0,p),p)\\
&\qquad\qquad -G(s,(\nu_\tau,\upsilon_\tau)(s;\xi_0,p_0),p_0)\|\Delta s\Big\}\quad\text{for }t\in\mathbb{T}^+_\tau\quad\text{by (3.2)}.\end{aligned}\tag{3.26}$$

Subtraction and addition of the expressions $\|F(s,(\nu_\tau,\upsilon_\tau)(s;\xi_0,p_0),p)\|$ and $\|G(s,(\nu_\tau,\upsilon_\tau)(s;\xi_0,p_0),p)\|$, respectively, lead to

$$\left\| \begin{pmatrix} \nu_\tau \\ \upsilon_\tau \end{pmatrix} (t;\xi_0,p) - \begin{pmatrix} \nu_\tau \\ \upsilon_\tau \end{pmatrix} (t;\xi_0,p_0) \right\| \le \max\{\alpha+\beta,\gamma+\delta\} \quad \text{for } t \in \mathbb{T}_\tau^+, \tag{3.27}$$

where (cf. (3.4))

$$\begin{aligned}
\alpha &:= K_1 \int_\tau^t e_a(t,\sigma(s)) \| F(s,(\nu_\tau,\upsilon_\tau)(s;\xi_0,p_0),p) - F(s,(\nu_\tau,\upsilon_\tau)(s;\xi_0,p_0),p_0) \| \Delta s, \\
\beta &:= K_1 |F|_1 \int_\tau^t e_a(t,\sigma(s)) \left\| \begin{pmatrix} \nu_\tau \\ \upsilon_\tau \end{pmatrix} (s;\xi_0,p) - \begin{pmatrix} \nu_\tau \\ \upsilon_\tau \end{pmatrix} (s;\xi_0,p_0) \right\| \Delta s, \\
\gamma &:= K_2 \int_t^\infty e_b(t,\sigma(s)) \| G(s,(\nu_\tau,\upsilon_\tau)(s;\xi_0,p_0),p) - G(s,(\nu_\tau,\upsilon_\tau)(s;\xi_0,p_0),p_0) \| \Delta s, \\
\delta &:= K_2 |G|_1 \int_t^\infty e_b(t,\sigma(s)) \left\| \begin{pmatrix} \nu_\tau \\ \upsilon_\tau \end{pmatrix} (s;\xi_0,p) - \begin{pmatrix} \nu_\tau \\ \upsilon_\tau \end{pmatrix} (s;\xi_0,p_0) \right\| \Delta s.
\end{aligned} \tag{3.28}$$

Now and in the further progress of this proof, we often use the elementary relation

$$\max\{\alpha+\beta,\gamma+\delta\} \le \alpha+\gamma+\max\{\beta,\delta\}, \tag{3.29}$$

which is valid for arbitrary reals $\alpha,\beta,\gamma,\delta \ge 0$, and obtain the estimate

$$\begin{aligned}
&\left\| \begin{pmatrix} \nu_\tau \\ \upsilon_\tau \end{pmatrix} (t;\xi_0,p) - \begin{pmatrix} \nu_\tau \\ \upsilon_\tau \end{pmatrix} (t;\xi_0,p_0) \right\| e_c(\tau,t) \\
&\quad \le \alpha e_c(\tau,t) + \gamma e_c(\tau,t) \\
&\qquad + \max\left\{ \frac{K_1|F|_1}{\lfloor c-a \rfloor}, \frac{K_2|G|_1}{\lfloor b-c \rfloor} \right\} \left\| \begin{pmatrix} \nu_\tau \\ \upsilon_\tau \end{pmatrix} (\xi_0,p) - \begin{pmatrix} \nu_\tau \\ \upsilon_\tau \end{pmatrix} (\xi_0,p_0) \right\|_{\tau,c}^+ \quad \text{for } t \in \mathbb{T}_\tau^+,
\end{aligned} \tag{3.30}$$

from [18, Lemma 1.3.29, page 65]. Hence, by passing over to the least upper bound for $t \in \mathbb{T}_\tau^+$, we get (cf. (3.15))

$$\left\| \begin{pmatrix} \nu_\tau \\ \upsilon_\tau \end{pmatrix} (\xi_0,p) - \begin{pmatrix} \nu_\tau \\ \upsilon_\tau \end{pmatrix} (\xi_0,p_0) \right\|_{\tau,c}^+ \le \frac{\max\{K_1,K_2\}}{1-L} \sup_{\tau \preceq t} U(t,p) \quad \text{by (3.24)} \tag{3.31}$$

with the mapping

$$\begin{aligned}
U(t,p) := e_c(\tau,t) &\int_\tau^t e_a(t,\sigma(s)) \| F(s,(\nu_\tau,\upsilon_\tau)(s;\xi_0,p_0),p) - F(s,(\nu_\tau,\upsilon_\tau)(s;\xi_0,p_0),p_0) \| \Delta s \\
&+ e_c(\tau,t) \int_t^\infty e_b(t,\sigma(s)) \| G(s,(\nu_\tau,\upsilon_\tau)(s;\xi_0,p_0),p) - G(s,(\nu_\tau,\upsilon_\tau)(s;\xi_0,p_0),p_0) \| \Delta s.
\end{aligned} \tag{3.32}$$

Therefore, it turns out to be sufficient to prove

$$\lim_{p\to p_0}\sup_{\tau\preceq t} U(t,p)=0 \tag{3.33}$$

to show the limit relation (3.25). We proceed indirectly. Assume (3.33) does not hold. Then there exist an $\epsilon>0$ and a sequence $(p_i)_{i\in\mathbb{N}}$ in $\mathscr{P}$ with $\lim_{i\to\infty} p_i=p_0$ and $\sup_{\tau\preceq t} U(t, p_i)>\epsilon$ for $i\in\mathbb{N}$. This implies the existence of a sequence $(t_i)_{i\in\mathbb{N}}$ in $\mathbb{T}_\tau^+$ such that

$$U(t_i,p_i)>\epsilon \quad \text{for } i\in\mathbb{N}. \tag{3.34}$$

From now on, we consider $a+\sigma\lhd c$, choose a fixed growth rate $d\in\mathscr{C}_{\mathrm{rd}}^+\mathscr{R}(\mathbb{T},\mathbb{R})$, $a+\sigma\lhd d\lhd c$, and remark that the inequality $d\lhd c$ will play an important role below. Because of Hypothesis 3.1(ii) and the inclusion $(\nu_\tau,\upsilon_\tau)(\xi_0,p)\in\mathscr{B}_{\tau,d}^+(\mathscr{X}\times\mathscr{Y})$, we get (cf. (3.4))

$$\begin{aligned}
\|F(s,(\nu_\tau,\upsilon_\tau)(s;\xi_0,p_0),p)\| &\le |F|_1\left\|\begin{pmatrix}\nu_\tau\\ \upsilon_\tau\end{pmatrix}(\xi_0,p_0)\right\|_{\tau,d}^+ e_d(s,\tau) \quad \text{for } s\in\mathbb{T}_\tau^+, \quad \text{by (3.3)}\\
\|G(s,(\nu_\tau,\upsilon_\tau)(s;\xi_0,p_0),p)\| &\le |G|_1\left\|\begin{pmatrix}\nu_\tau\\ \upsilon_\tau\end{pmatrix}(\xi_0,p_0)\right\|_{\tau,d}^+ e_d(s,\tau) \quad \text{for } s\in\mathbb{T}_\tau^+, \quad \text{by (3.3)}
\end{aligned} \tag{3.35}$$

and the triangle inequality leads to

$$\begin{aligned}
U(t,p) &\le 2|F|_1\left\|\begin{pmatrix}\nu_\tau\\ \upsilon_\tau\end{pmatrix}(\xi_0,p_0)\right\|_{\tau,d}^+ e_c(\tau,t)\int_\tau^t e_a(t,\sigma(s))e_d(s,\tau)\Delta s\\
&\quad+2|G|_1\left\|\begin{pmatrix}\nu_\tau\\ \upsilon_\tau\end{pmatrix}(\xi_0,p_0)\right\|_{\tau,d}^+ e_c(\tau,t)\int_t^\infty e_b(t,\sigma(s))e_d(s,\tau)\Delta s \quad \text{by (3.32)}\\
&\le 2\max\{|F|_1,|G|_1\}\left\|\begin{pmatrix}\nu_\tau\\ \upsilon_\tau\end{pmatrix}(\xi_0,p_0)\right\|_{\tau,d}^+\left(\frac{1}{\lfloor d-a\rfloor}+\frac{1}{\lfloor b-d\rfloor}\right)e_{d\ominus c}(t,\tau) \quad \text{for } t\in\mathbb{T}_\tau^+,
\end{aligned} \tag{3.36}$$

where we have evaluated the integrals using [17, Lemma 1.3.29, page 65]. Because of $d\lhd c$ and [17, Lemma 1.3.26, page 63], passing over to the limit $t\to\infty$ yields $\lim_{t\to\infty}U(t,p)=0$ uniformly in $p\in\mathscr{P}$, and taking into account (3.34), the sequence $(t_i)_{i\in\mathbb{N}}$ in $\mathbb{T}_\tau^+$ has to be bounded above, that is, there exists a time $T\in(\tau,\infty)_{\mathbb{T}}$ with $t_i\preceq T$ for all $i\in\mathbb{N}$. Hence, by [9, Theorem 7.4(i)], we can deduce

$$\begin{aligned}
&U(t_i,p_i)\\
&\le\int_\tau^T e_c(\tau,\sigma(s))\|F(s,(\nu_\tau,\upsilon_\tau)(s;\xi_0,p_0),p_i)-F(s,(\nu_\tau,\upsilon_\tau)(s;\xi_0,p_0),p_0)\|\Delta s\\
&\qquad+\int_\tau^\infty e_c(\tau,\sigma(s))e_{b\ominus c}(T,\sigma(s))\|G(s,(\nu_\tau,\upsilon_\tau)(s;\xi_0,p_0),p_i)\\
&\qquad\qquad-G(s,(\nu_\tau,\upsilon_\tau)(s;\xi_0,p_0),p_0)\|\Delta s \quad \text{by (3.32)}
\end{aligned} \tag{3.37}$$

for $i \in \mathbb{N}$, where the first finite integral tends to zero for $i \to \infty$ by the continuity of F. Continuity of G implies $\lim_{i\to\infty} G(s,(\nu_\tau,\upsilon_\tau)(s;\xi_0,p_0),p_i) = G(s,(\nu_\tau,\upsilon_\tau)(s;\xi_0,p_0),p_0)$ and with the Lebesgue's theorem (here, one has to apply the Lipschitz estimate for the mapping G, which is implied by (3.4), to see that the function

$$s \mapsto e_c(s,\sigma(s))e_{b\ominus c}(T,\sigma(s))|G|_1\|(\nu_\tau,\upsilon_\tau)(\xi_0,p_0)\|_{\tau,c}^+ \tag{3.38}$$

is an integrable majorant) for the integral on $\mathbb{T}$ (cf. [16, Nr. 313, page 161]), we get the convergence of the indefinite integral to zero for $i \to \infty$. Thus we derived the relation $\lim_{i\to\infty} U(t_i,p_i) = 0$, which obviously contradicts (3.34). Up to now we have shown the continuity of $(\nu_\tau,\upsilon_\tau)(\xi_0,\cdot) : \mathcal{P} \to \mathcal{B}_{\tau,c}^+(\mathcal{X}\times\mathcal{Y})$, and Lemma 3.4(b) gives us the Lipschitz estimate

$$\left\| \begin{pmatrix}\nu_\tau\\ \upsilon_\tau\end{pmatrix}(\xi,p_0) - \begin{pmatrix}\nu_\tau\\ \upsilon_\tau\end{pmatrix}(\xi_0,p_0)\right\|_{\tau,c}^+ \le \frac{K_1}{1-L}\|\xi-\xi_0\| \quad \text{by (3.16)} \tag{3.39}$$

for any $\xi \in \mathcal{X}$. So, for example, [3, Lemma B.4] implies the desired continuity of the fixed point mapping $(\nu_\tau,\upsilon_\tau) : \mathcal{X}\times\mathcal{P} \to \mathcal{B}_{\tau,c}^+(\mathcal{X}\times\mathcal{Y})$. By properties of the evaluation map (see [18, Lemma 3.4]), this yields also that $(\nu_\tau,\upsilon_\tau)(t;\cdot) : \mathcal{X}\times\mathcal{P} \to \mathcal{X}\times\mathcal{Y}$, $t \in \mathbb{T}_\tau^+$, is continuous.

Step 2. Claim: *the mapping* $s : \mathbb{T}\times\mathcal{X}\times\mathcal{P} \to \mathcal{Y}$ *is continuous.*

Let $\tau_0 \in \mathbb{T}$, $\xi_0 \in \mathcal{X}$, and $p_0 \in \mathcal{P}$ be fixed. From (3.25) and the definition of s, we have

$$\lim_{p\to p_0} s(\tau_0,\xi_0,p) = s(\tau_0,\xi_0,p_0), \tag{3.40}$$

and, similarly, (3.39) leads to the estimate

$$\begin{aligned}&\|s(\tau,\xi,p) - s(\tau_0,\xi_0,p_0)\|\\ &\quad\le \frac{K_1}{1-L}\|\xi-\xi_0\| + \|s(\tau,\xi_0,p) - s(\tau_0,\xi_0,p)\| + \|s(\tau_0,\xi_0,p) - s(\tau_0,\xi_0,p_0)\|\end{aligned} \tag{3.41}$$

for $\tau \in \mathbb{T}$, $\xi \in \mathcal{X}$, and $p \in \mathcal{P}$. Therefore, to establish the claim of Step 2, it remains to show the limit relation

$$\lim_{\tau\to\tau_0} s(\tau,\xi_0,p) = s(\tau_0,\xi_0,p) \quad \text{uniformly in } p \in \mathcal{P}. \tag{3.42}$$

We abbreviate $\phi(\tau,p) := (\phi_1,\phi_2)(\tau,p) := \varphi(\tau;\tau_0,\xi_0,s(\tau_0,\xi_0,p),p)$ and by Theorem 2.4(a), $\phi(\cdot,p)$ exists in a $\mathbb{T}$-neighborhood of τ_0 independent of $p \in \mathcal{P}$. The invariance of $S(p)$, $p \in \mathcal{P}$, implies $\phi_2(\tau,p) = s(\tau,\phi_1(\tau,p),p)$, as well as $\phi_1(\tau_0,p) = \xi_0$, $\phi_2(\tau_0,p) = s(\tau_0,\xi_0,p)$.

Hence, one obtains

$$\begin{aligned}
&\|s(\tau,\xi_0,p)-s(\tau_0,\xi_0,p)\| \\
&\quad\le \|s(\tau,\xi_0,p)-s(\tau_0,\phi_1(\tau,p),p)\|+\|s(\tau,\phi_1(\tau,p),p)-s(\tau_0,\xi_0,p)\| \\
&\quad\le \frac{K_1}{1-L}\|\phi_1(\tau_0,p)-\phi_1(\tau,p)\|+\|\phi_2(\tau,p)-\phi_2(\tau_0,p)\| \quad \text{by (3.39)} \\
&\quad\le \left(\frac{K_1}{1-L}+1\right)\|\phi(\tau,p)-\phi(\tau_0,p)\| \quad \text{for } p\in\mathscr{P} \quad \text{by (2.1)},
\end{aligned} \tag{3.43}$$

and, because of (a_1), it is

$$\begin{aligned}
\|\phi(\tau_0,p)\| &\le \max\{\|\xi_0\|,\|s(\tau_0,\xi_0,p)-s(\tau_0,0,p)\|\} \quad \text{by (2.1)} \\
&\le \max\left\{1,\frac{K_1}{1-L}\right\}\|\xi_0\| \quad \text{for } p\in\mathscr{P} \quad \text{by (3.39)}.
\end{aligned} \tag{3.44}$$

Consequently, we can apply Theorem 2.4(b) (with $\xi(p)=\phi(\tau_0,p)$) and get

$$\lim_{\tau\to\tau_0}\phi(\tau,p)=\phi(\tau_0,p) \quad \text{uniformly in } p\in\mathscr{P}, \tag{3.45}$$

which ultimately guarantees (3.42).

Step 3. Let $c\in\mathscr{C}^+_{\mathrm{rd}}\mathscr{R}(\mathbb{T},\mathbb{R})$, $a+\sigma\trianglelefteq c\trianglelefteq b-\sigma$, $\xi\in\mathscr{X}$, and $p\in\mathscr{P}$ be arbitrary. By formal differentiation of the fixed point equation (cf. (3.14), (3.17))

$$\begin{pmatrix}\nu_\tau\\ \upsilon_\tau\end{pmatrix}(t;\xi,p)=\begin{pmatrix}\Phi_A(t,\tau)\xi+\int_\tau^t\Phi_A(t,\sigma(s))F(s,(\nu_\tau,\upsilon_\tau)(s;\xi,p),p)\Delta s\\ -\int_t^\infty\Phi_B(t,\sigma(s))G(s,(\nu_\tau,\upsilon_\tau)(s;\xi,p),p)\Delta s\end{pmatrix} \quad \text{for } t\in\mathbb{T}^+_\tau, \tag{3.46}$$

with respect to $\xi\in\mathscr{X}$, we obtain another fixed point equation

$$\begin{pmatrix}\nu^1_\tau\\ \upsilon^1_\tau\end{pmatrix}(\xi,p)=\mathscr{T}^1_\tau((\nu^1_\tau,\upsilon^1_\tau)(\xi,p);\xi,p) \tag{3.47}$$

for the formal partial derivative $(\nu^1_\tau,\upsilon^1_\tau)$ of $(\nu_\tau,\upsilon_\tau):\mathscr{X}\times\mathscr{P}\to\mathscr{B}^+_{\tau,c}(\mathscr{X}\times\mathscr{Y})$ with respect to $\xi\in\mathscr{X}$, where the right-hand side of (3.47) is given by

$$\begin{aligned}
&\mathscr{T}^1_\tau(\nu^1,\upsilon^1;\xi,p) \\
&\quad:=\begin{pmatrix}\Phi_A(\cdot,\tau)+\int_\tau^\cdot\Phi_A(\cdot,\sigma(s))D_{(2,3)}F(s,(\nu_\tau,\upsilon_\tau)(s;\xi,p),p)\begin{pmatrix}\nu^1\\ \upsilon^1\end{pmatrix}(s)\Delta s\\ -\int_\cdot^\infty\Phi_B(\cdot,\sigma(s))D_{(2,3)}G(s,(\nu_\tau,\upsilon_\tau)(s;\xi,p),p)\begin{pmatrix}\nu^1\\ \upsilon^1\end{pmatrix}(s)\Delta s\end{pmatrix}.
\end{aligned} \tag{3.48}$$

Here, (ν^1,υ^1) is a mapping from $\mathbb{T}^+_\tau$ to $\mathscr{L}(\mathscr{X};\mathscr{X}\times\mathscr{Y})$ and in the following we investigate this operator $\mathscr{T}^1_\tau$.

Step 4. Claim: *for every growth rate $c \in \mathscr{C}_{rd}^{+}\mathscr{R}(\mathbb{T},\mathbb{R})$, $a+\sigma \trianglelefteq c \trianglelefteq b-\sigma$, the operator $\mathscr{T}_\tau^1 : \mathscr{B}_{\tau,c}^1 \times \mathscr{X} \times \mathscr{P} \to \mathscr{B}_{\tau,c}^1$ is well defined and satisfies the estimate*

$$\|\mathscr{T}_\tau^1(\nu^1,v^1;\xi,p)\|_{\tau,c}^{+} \leq K_1 + L\left\|\begin{pmatrix}\nu^1\\ v^1\end{pmatrix}\right\|_{\tau,c}^{+} \quad \text{for } (\nu^1,v^1) \in \mathscr{B}_{\tau,c}^1,\ \xi \in \mathscr{X},\ p \in \mathscr{P}. \tag{3.49}$$

Thereto choose arbitrary functions $(\nu^1,v^1) \in \mathscr{B}_{\tau,c}^1$ and $\xi \in \mathscr{X}$, $p \in \mathscr{P}$. Now using (3.2), (3.4), and [17, Lemma 1.3.29, page 65], it is

$$\begin{aligned}
&\|\mathscr{T}_\tau^1(\nu^1,v^1;\xi,p)(t)\|_{\mathscr{L}(\mathscr{X};\mathscr{X}\times\mathscr{Y})}e_c(\tau,t)\\
&\quad\leq \max\Bigg\{K_1e_{c\ominus a}(\tau,t)+K_1|F|_1e_c(\tau,t)\int_\tau^t e_a(t,\sigma(s))\left\|\begin{pmatrix}\nu^1\\ v^1\end{pmatrix}(s)\right\|\Delta s,\\
&\qquad\qquad K_2|G|_1e_c(\tau,t)\int_t^\infty e_b(t,\sigma(s))\left\|\begin{pmatrix}\nu^1\\ v^1\end{pmatrix}(s)\right\|\Delta s\Bigg\} \quad \text{by (3.48)}\\
&\quad\leq K_1+e_c(\tau,t)\max\Bigg\{K_1|F|_1\int_\tau^t e_a(t,\sigma(s))e_c(s,\tau)\Delta s,\\
&\qquad\qquad K_2|G|_1\int_t^\infty e_b(t,\sigma(s))e_c(s,\tau)\Delta s\Bigg\}\left\|\begin{pmatrix}\nu^1\\ v^1\end{pmatrix}\right\|_{\tau,c}^{+} \quad \text{by (3.29)}\\
&\quad\leq K_1+\max\left\{\frac{K_1|F|_1}{\lfloor c-a\rfloor},\frac{K_2|G|_1}{\lfloor b-c\rfloor}\right\}\left\|\begin{pmatrix}\nu^1\\ v^1\end{pmatrix}\right\|_{\tau,c}^{+}\\
&\quad\leq K_1+L\left\|\begin{pmatrix}\nu^1\\ v^1\end{pmatrix}\right\|_{\tau,c}^{+} \quad \text{for } t\in\mathbb{T}_\tau^{+} \quad \text{by (3.24)},
\end{aligned} \tag{3.50}$$

and passing over to the least upper bound over $t \in \mathbb{T}_\tau^{+}$ implies our claim $\mathscr{T}_\tau^1(\nu^1,v^1;\xi,p) \in \mathscr{B}_{\tau,c}^1$, as well as the estimate (3.49).

Step 5. Claim: *for every growth rate $c \in \mathscr{C}_{rd}^{+}\mathscr{R}(\mathbb{T},\mathbb{R})$, $a+\sigma \trianglelefteq c \trianglelefteq b-\sigma$, the operator $\mathscr{T}_\tau^1(\cdot;\xi,p) : \mathscr{B}_{\tau,c}^1 \to \mathscr{B}_{\tau,c}^1$ is a uniform contraction in $\xi \in \mathscr{X}$, $p \in \mathscr{P}$; moreover, the fixed point $(\nu_\tau^1,v_\tau^1)(\xi,p) \in \mathscr{B}_{\tau,c}^1$ does not depend on $c \in \mathscr{C}_{rd}^{+}\mathscr{R}(\mathbb{T},\mathbb{R})$, $a+\sigma \trianglelefteq c \trianglelefteq b-\sigma$, and satisfies*

$$\left\|\begin{pmatrix}\nu_\tau^1\\ v_\tau^1\end{pmatrix}(\xi,p)\right\|_{\tau,c}^{+} \leq \frac{K_1}{1-L} \quad \text{for } \xi \in \mathscr{X},\ p \in \mathscr{P}. \tag{3.51}$$

Let $\xi \in \mathscr{X}$ and $p \in \mathscr{P}$ be arbitrary. Completely analogous to the estimate (3.50), we get

$$\begin{aligned}
&\|\mathscr{T}_\tau^1(\nu^1,v^1;\xi,p)-\mathscr{T}_\tau^1(\bar\nu^1,\bar v^1;\xi,p)\|_{\tau,c}^{+}\\
&\quad\leq L\left\|\begin{pmatrix}\nu^1\\ v^1\end{pmatrix}-\begin{pmatrix}\bar\nu^1\\ \bar v_1\end{pmatrix}\right\|_{\tau,c}^{+} \quad \text{for } (\nu^1,v^1),(\bar\nu^1,\bar v^1) \in \mathscr{B}_{\tau,c}^1 \quad \text{by (3.24)}.
\end{aligned} \tag{3.52}$$

Taking (3.15) into account, consequently Banach's fixed point theorem guarantees the unique existence of a fixed point $(\nu_\tau^1,v_\tau^1)(\xi,p) \in \mathscr{B}_{\tau,c}^1$ of $\mathscr{T}_\tau^1(\cdot;\xi,p) : \mathscr{B}_{\tau,c}^1 \to \mathscr{B}_{\tau,c}^1$. This fixed point is independent of the growth constant $c \in \mathscr{C}^+_{\text{rd}}\mathscr{R}(\mathbb{T},\mathbb{R})$, $a+\sigma \trianglelefteq c \trianglelefteq b-\sigma$, because with Lemma 2.3(b) and (c) we have the inclusion $\mathscr{B}^1_{\tau,a+\sigma} \subseteq \mathscr{B}^1_{\tau,c}$ and every mapping $\mathscr{T}^1_\tau(\cdot;\xi,p) : \mathscr{B}^1_{\tau,c} \to \mathscr{B}^1_{\tau,c}$ has the same fixed point as the restriction $\mathscr{T}^1_\tau(\cdot;\xi,p)|_{\mathscr{B}^1_{\tau,a+\sigma}}$. Finally the fixed point identity (3.47) and (3.49) leads to the estimate (3.51).

Step 6. Claim: *for every growth rate $c \in \mathscr{C}^+_{rd}\mathscr{R}(\mathbb{T},\mathbb{R})$, $a+\sigma \lhd c \trianglelefteq b-\sigma$, and $p \in \mathscr{P}$, the mapping $(\nu_\tau,\upsilon_\tau)(\cdot,p) : \mathscr{X} \to \mathscr{B}^+_{\tau,c}(\mathscr{X}\times\mathscr{Y})$ is differentiable with derivative*

$$D_1\begin{pmatrix}\nu_\tau\\ \upsilon_\tau\end{pmatrix} = \begin{pmatrix}\nu^1_\tau\\ \upsilon^1_\tau\end{pmatrix} : \mathscr{X}\times\mathscr{P} \longrightarrow \mathscr{B}^1_{\tau,c}. \tag{3.53}$$

Let $\xi \in \mathscr{X}$ and $p \in \mathscr{P}$ be arbitrary. In relation (3.53), as well as in the subsequent considerations, we are using the isomorphism between the spaces $\mathscr{B}^1_{\tau,c}$ and $\mathscr{L}(\mathscr{X};\mathscr{B}^+_{\tau,c}(\mathscr{X}\times\mathscr{Y}))$ from Lemma 2.3(c) and identify them. To show the claim above, we define the following four quotients:

$$\begin{aligned}\Delta\nu(s,h) &:= \frac{\nu_\tau(s;\xi+h,p)-\nu_\tau(s;\xi,p)-\nu^1_\tau(s;\xi,p)h}{\|h\|},\\ \Delta\upsilon(s,h) &:= \frac{\upsilon_\tau(s;\xi+h,p)-\upsilon_\tau(s;\xi,p)-\upsilon^1_\tau(s;\xi,p)h}{\|h\|},\end{aligned} \tag{3.54}$$

$$\begin{aligned}\Delta F(s,x,y,h_1,h_2) &:= \frac{F(s,x+h_1,y+h_2,p)-F(s,x,y,p)-D_{(2,3)}F(s,x,y,p)\begin{pmatrix}h_1\\ h_2\end{pmatrix}}{\|(h_1,h_2)\|},\\ \Delta G(s,x,y,h_1,h_2) &:= \frac{G(s,x+h_1,y+h_2,p)-G(s,x,y,p)-D_{(2,3)}G(s,x,y,p)\begin{pmatrix}h_1\\ h_2\end{pmatrix}}{\|(h_1,h_2)\|},\end{aligned} \tag{3.55}$$

for times $s \in \mathbb{T}$ and $x \in \mathscr{X}$, $h,h_1 \in \mathscr{X}\setminus\{0\}$, $y \in \mathscr{Y}$, $h_2 \in \mathscr{Y}\setminus\{0\}$. Thereby obviously the inclusion $(\Delta\nu,\Delta\upsilon)(\cdot,h) \in \mathscr{B}^+_{\tau,c}(\mathscr{X}\times\mathscr{Y})$ holds. To prove the differentiability we have to show the limit relation

$$\lim_{h\to 0}\begin{pmatrix}\Delta\nu\\ \Delta\upsilon\end{pmatrix}(\cdot,h) = 0 \quad \text{in } \mathscr{B}^+_{\tau,c}(\mathscr{X}\times\mathscr{Y}). \tag{3.56}$$

For this, consider $a+\sigma \lhd c$, a growth rate $d \in \mathscr{C}^+_{\text{rd}}\mathscr{R}(\mathbb{T},\mathbb{R})$, $a+\sigma \lhd d \lhd c$, and from Lemma 3.3, we obtain

$$\begin{aligned}&\frac{1}{\|h\|}\left\|\begin{pmatrix}\nu_\tau\\ \upsilon_\tau\end{pmatrix}(s;\xi+h,p)-\begin{pmatrix}\nu_\tau\\ \upsilon_\tau\end{pmatrix}(s;\xi,p)\right\|\\ &\qquad\le K_1\frac{\lfloor d-a\rfloor}{\lfloor d-a\rfloor - K_1|F|_1}e_d(s,\tau) \quad \text{for } s\in\mathbb{T}^+_\tau \quad \text{by (3.6)}.\end{aligned} \tag{3.57}$$

Moreover, using the fixed point equations (3.46) for ν_τ and (3.47) for ν_τ^1, it results (cf. (3.14), (3.48)) that

$$\begin{aligned} \|\Delta\nu(t,h)\| = \frac{1}{\|h\|}\Bigg\| \int_\tau^t \Phi_A(t,\sigma(s)) & \\ \times \bigg[F(s,(\nu_\tau,\upsilon_\tau)(s;\xi+h,p),p) - F(s,(\nu_\tau,\upsilon_\tau)(s;\xi,p),p) & \\ - D_{(2,3)}F(s,(\nu_\tau,\upsilon_\tau)(s;\xi,p),p)\begin{pmatrix}\nu_\tau^1\\ \upsilon_\tau^1\end{pmatrix}(s;\xi,p)h\bigg]\Delta s\Bigg\| & \quad \text{for } t\in\mathbb{T}_\tau^+, \end{aligned} \tag{3.58}$$

where subtraction and addition of the expression

$$D_{(2,3)}F(s,(\nu_\tau,\upsilon_\tau)(s;\xi,p),p)\left[\begin{pmatrix}\nu_\tau\\ \upsilon_\tau\end{pmatrix}(s;\xi+h,p) - \begin{pmatrix}\nu_\tau\\ \upsilon_\tau\end{pmatrix}(s;\xi,p)\right] \tag{3.59}$$

in the above brackets imply the estimate

$$\begin{aligned} \|\Delta\nu(t,h)\| \le{}& \frac{1}{\|h\|}\Bigg\|\int_\tau^t \Phi_A(t,\sigma(s))\bigg\{F(s,(\nu_\tau,\upsilon_\tau)(s;\xi+h,p),p) - F(s,(\nu_\tau,\upsilon_\tau)(s;\xi,p),p) \\ &\qquad - D_{(2,3)}F(s,(\nu_\tau,\upsilon_\tau)(s;\xi,p),p) \\ &\qquad \times\left[\begin{pmatrix}\nu_\tau\\ \upsilon_\tau\end{pmatrix}(s;\xi+h,p) - \begin{pmatrix}\nu_\tau\\ \upsilon_\tau\end{pmatrix}(s;\xi,p)\right]\bigg\}\Delta s\Bigg\| \\ &+ \frac{1}{\|h\|}\Bigg\|\int_\tau^t \Phi_A(t,\sigma(s))D_{(2,3)}F(s,(\nu_\tau,\upsilon_\tau)(s;\xi,p),p) \\ &\qquad \cdot\left[\begin{pmatrix}\nu_\tau\\ \upsilon_\tau\end{pmatrix}(s;\xi+h,p) - \begin{pmatrix}\nu_\tau\\ \upsilon_\tau\end{pmatrix}(s;\xi,p) - \begin{pmatrix}\nu_\tau^1\\ \upsilon_\tau^1\end{pmatrix}(s;\xi,p)h\right]\Delta s\Bigg\| \\ \le{}& \int_\tau^t \|\Phi_A(t,\sigma(s))\|\|\Delta F(s,(\nu_\tau,\upsilon_\tau)(s;\xi,p),(\nu_\tau,\upsilon_\tau)(s;\xi+h,p)-(\nu_\tau,\upsilon_\tau)(s;\xi,p))\| \\ &\qquad \cdot\frac{1}{\|h\|}\left\|\begin{pmatrix}\nu_\tau\\ \upsilon_\tau\end{pmatrix}(s;\xi+h,p) - \begin{pmatrix}\nu_\tau\\ \upsilon_\tau\end{pmatrix}(s;\xi,p)\right\|\Delta s \\ &+ |F|_1\int_\tau^t \|\Phi_A(t,\sigma(s))\|\left\|\begin{pmatrix}\Delta\nu\\ \Delta\upsilon\end{pmatrix}(s,h)\right\|\Delta s \quad \text{by (3.4)} \\ \le{}& K_1\int_\tau^t e_a(t,\sigma(s))\|\Delta F(s,(\nu_\tau,\upsilon_\tau)(s;\xi,p),(\nu_\tau,\upsilon_\tau)(s;\xi+h,p)-(\nu_\tau,\upsilon_\tau)(s;\xi,p))\| \\ &\qquad \cdot\frac{1}{\|h\|}\left\|\begin{pmatrix}\nu_\tau\\ \upsilon_\tau\end{pmatrix}(s;\xi+h,p) - \begin{pmatrix}\nu_\tau\\ \upsilon_\tau\end{pmatrix}(s;\xi,p)\right\|\Delta s \\ &+ K_1|F|_1\int_\tau^t e_a(t,\sigma(s))\left\|\begin{pmatrix}\Delta\nu\\ \Delta\upsilon\end{pmatrix}(s,h)\right\|\Delta s \quad \text{by (3.2)} \end{aligned} \tag{3.60}$$

for $t \in \mathbb{T}_\tau^+$, and together with (3.57), we get

$$\begin{aligned}
&\|\Delta\nu(t,h)\| \\
&\quad\le K_1|F|_1\int_\tau^t e_a(t,\sigma(s))\left\|\begin{pmatrix}\Delta\nu\\ \Delta\upsilon\end{pmatrix}(s,h)\right\|\Delta s+\frac{K_1^2\lfloor d-a\rfloor}{\lfloor d-a\rfloor-K_1|F|_1} \\
&\qquad\cdot\int_\tau^t e_a(t,\sigma(s))e_d(s,\tau)\|\Delta F(s,(\nu_\tau,\upsilon_\tau)(s;\xi,p),(\nu_\tau,\upsilon_\tau)(s;\xi+h,p)-(\nu_\tau,\upsilon_\tau)(s;\xi,p))\|\Delta s
\end{aligned} \tag{3.61}$$

for $t \in \mathbb{T}_\tau^+$. Now we analogously derive a similar estimate for the norm of the second component $\|\Delta\upsilon(t,h)\|$ and obtain

$$\begin{aligned}
&\|\Delta\upsilon(t,h)\| \\
&\quad\le K_2|G|_1\int_t^\infty e_b(t,\sigma(s))\left\|\begin{pmatrix}\Delta\nu\\ \Delta\upsilon\end{pmatrix}(s,h)\right\|\Delta s+\frac{K_1K_2\lfloor d-a\rfloor}{\lfloor d-a\rfloor-K_1|F|_1} \\
&\qquad\cdot\int_t^\infty e_b(t,\sigma(s))e_d(s,\tau)\|\Delta G(s,(\nu_\tau,\upsilon_\tau)(s;\xi,p),(\nu_\tau,\upsilon_\tau)(s;\xi+h,p)-(\nu_\tau,\upsilon_\tau)(s;\xi,p))\|\Delta s
\end{aligned} \tag{3.62}$$

for $t \in \mathbb{T}_\tau^+$. Consequently, for the norm $\|(\Delta\nu,\Delta\upsilon)(t,h)\|$, one gets the inequality

$$\begin{aligned}
\left\|\begin{pmatrix}\Delta\nu\\ \Delta\upsilon\end{pmatrix}(t,h)\right\| &= \max\{\|\Delta\nu(t,h)\|,\|\Delta\upsilon(t,h)\|\} \quad \text{by (2.1)} \\
&\le \max\{\alpha+\beta,\gamma+\delta\} \quad \text{for } t\in\mathbb{T}_\tau^+,
\end{aligned} \tag{3.63}$$

with

$$\begin{aligned}
\alpha &:= \frac{K_1^2\lfloor d-a\rfloor}{\lfloor d-a\rfloor-K_1|F|_1} \\
&\qquad\cdot\int_\tau^t e_a(t,\sigma(s))e_d(s,\tau)\|\Delta F(s,(\nu_\tau,\upsilon_\tau)(s;\xi,p),(\nu_\tau,\upsilon_\tau)(s;\xi+h,p)-(\nu_\tau,\upsilon_\tau)(s;\xi,p))\|\Delta s, \\
\beta &:= K_1|F|_1\int_\tau^t e_a(t,\sigma(s))\left\|\begin{pmatrix}\Delta\nu\\ \Delta\upsilon\end{pmatrix}(s,h)\right\|\Delta s, \\
\gamma &:= \frac{K_1K_2\lfloor d-a\rfloor}{\lfloor d-a\rfloor-K_1|F|_1} \\
&\qquad\cdot\int_t^\infty e_b(t,\sigma(s))e_d(s,\tau)\|\Delta G(s,(\nu_\tau,\upsilon_\tau)(s;\xi,p),(\nu_\tau,\upsilon_\tau)(s;\xi+h,p)-(\nu_\tau,\upsilon_\tau)(s;\xi,p))\|\Delta s, \\
\delta &:= K_2|G|_1\int_t^\infty e_b(t,\sigma(s))\left\|\begin{pmatrix}\Delta\nu\\ \Delta\upsilon\end{pmatrix}(s,h)\right\|\Delta s.
\end{aligned} \tag{3.64}$$

We are using relation (3.29) again, and obtain the estimate (cf. [17, Lemma 1.3.29, page 65])

$$\left\| \begin{pmatrix} \Delta\nu \\ \Delta\upsilon \end{pmatrix} (t,h) \right\| e_c(\tau,t) \leq \alpha e_c(\tau,t) + \gamma e_c(\tau,t) + L \left\| \begin{pmatrix} \Delta\nu \\ \Delta\upsilon \end{pmatrix} (h) \right\|_{\tau,c}^{+} \quad \text{for } t \in \mathbb{T}_\tau^+ \quad \text{by (3.24)}. \tag{3.65}$$

By passing over to the least upper bound for $t \in \mathbb{T}_\tau^+$, we get (cf. (3.15))

$$\left\| \begin{pmatrix} \Delta\nu \\ \Delta\upsilon \end{pmatrix} (h) \right\|_{\tau,c}^{+} \leq \frac{K_1 \max\{K_1, K_2\}}{1-L} \frac{\lfloor d-a \rfloor}{\lfloor d-a \rfloor - K_1 |F|_1} \sup_{\tau \preceq t} V(t,h) \tag{3.66}$$

with

$$\begin{aligned} V(t,h) := e_c(\tau,t) \int_\tau^t & e_a(t,\sigma(s)) e_d(s,\tau) \| \Delta F(s, (\nu_\tau, \upsilon_\tau)(s;\xi,p), (\nu_\tau, \upsilon_\tau)(s;\xi+h,p) \\ & - (\nu_\tau, \upsilon_\tau)(s;\xi,p)) \| \Delta s \\ + e_c(\tau,t) \int_t^\infty & e_b(\sigma(s)) e_d(s,\tau) \| \Delta G(s, (\nu_\tau, \upsilon_\tau)(s;\xi,p), (\nu_\tau, \upsilon_\tau)(s;\xi+h,p) \\ & - (\nu_\tau, \upsilon_\tau)(s;\xi,p)) \| \Delta s \end{aligned} \tag{3.67}$$

for $t \in \mathbb{T}_\tau^+$. Thus, to prove the above claim in Step 6, we only have to show the limit relation

$$\lim_{h \to 0} \sup_{\tau \preceq t} V(t,h) = 0, \tag{3.68}$$

which will be done indirectly. Suppose (3.68) is not true. Then there exist an $\epsilon > 0$ and a sequence $(h_i)_{i \in \mathbb{N}}$ in $\mathscr{X}$ with $\lim_{i \to \infty} h_i = 0$ such that $\sup_{\tau \preceq t} V(t,h_i) > \epsilon$ for $i \in \mathbb{N}$. This implies the existence of a further sequence $(t_i)_{i \in \mathbb{N}}$ in $\mathbb{T}_\tau^+$ with

$$V(t_i,h_i) > \epsilon \quad \text{for } i \in \mathbb{N}. \tag{3.69}$$

Using the estimates $\|\Delta F(s,x,y,h_1,h_2)\| \leq 2|F|_1$ and $\|\Delta G(s,x,y,h_1,h_2)\| \leq 2|G|_1$, which result from (3.4) in connection with [14, Corollary 4.3, page 342], it follows by known arguments that

$$\begin{aligned} & V(t,h) \\ & \leq 2|F|_1 e_c(\tau,t) \int_\tau^t e_a(t,\sigma(s)) e_d(s,\tau) \Delta s + 2|G|_1 e_c(\tau,t) \int_t^\infty e_b(t,\sigma(s)) e_d(s,\tau) \Delta s \quad \text{by (3.67)} \\ & \leq \left(\frac{2|F|_1}{\lfloor d-a \rfloor} + \frac{2|G|_1}{\lfloor b-d \rfloor} \right) e_{d \ominus c}(t,\tau) \quad \text{for } t \in \mathbb{T}_\tau^+, \end{aligned} \tag{3.70}$$

and the right-hand side of this estimate converges to 0 for $t \to \infty$, that is, we have $\lim_{t\to\infty} V(t,h) = 0$ uniformly in $h \in \mathscr{X}$. Because of (3.69), the sequence $(t_i)_{i\in\mathbb{N}}$ has to be bounded in $\mathbb{T}_\tau^+$, that is, there exists a time $T \in (\tau,\infty)_{\mathbb{T}}$ with $t_i \preceq T$ for any $i \in \mathbb{N}$. Now we obtain

$$\begin{aligned}
&V(t_i,h_i)\\
&\le \int_\tau^T e_c(\tau,\sigma(s))e_d(s,\tau)\|\Delta F(s,(\nu_\tau,v_\tau)(s;\xi,p),(\nu_\tau,v_\tau)(s;\xi+h_i,p)-(\nu_\tau,v_\tau)(s;\xi,p))\|\Delta s\\
&\quad + \int_\tau^\infty e_c(\tau,\sigma(s))e_{b\ominus c}(T,\sigma(s))e_d(s,\tau)\\
&\qquad \cdot \|\Delta G(s,(\nu_\tau,v_\tau)(s;\xi,p),(\nu_\tau,v_\tau)(s;\xi+h_i,p)-(\nu_\tau,v_\tau)(s;\xi,p))\|\Delta s \quad \text{for } i \in \mathbb{N},
\end{aligned} \tag{3.71}$$

by (3.67) and because of Step 1, we have

$$\lim_{i\to\infty} \begin{pmatrix}\nu_\tau\\ v_\tau\end{pmatrix}(s;\xi+h_i,p) = \begin{pmatrix}\nu_\tau\\ v_\tau\end{pmatrix}(s;\xi,p) \quad \text{for } s \in \mathbb{T}_\tau^+,\ \xi \in \mathscr{X},\ p \in \mathscr{P}, \tag{3.72}$$

as well as, using the partial differentiability of F and G,

$$\lim_{(h_1,h_2)\to(0,0)} \left\| \begin{pmatrix}\Delta F\\ \Delta G\end{pmatrix}(s,x,y,h_1,h_2)\right\| = 0 \quad \text{for } x \in \mathscr{X},\ y \in \mathscr{Y}, \tag{3.73}$$

which leads to the limit relation

$$\lim_{i\to\infty} \left\| \begin{pmatrix}\Delta F\\ \Delta G\end{pmatrix}(s,(\nu_\tau,v_\tau)(s;\xi,p),(\nu_\tau,v_\tau)(s;\xi+h_i,p)-(\nu_\tau,v_\tau)(s;\xi,p))\right\| = 0 \quad \text{for } s \in \mathbb{T}_\tau^+. \tag{3.74}$$

Therefore the finite integral in (3.71) tends to 0 for $i \to \infty$. Using Lebesgue's theorem, also the indefinite integral in (3.71) converges to 0 for $i \to \infty$, and we finally have $\lim_{i\to\infty} V(t_i,h_i) = 0$, which contradicts (3.69). Hence the claim in Step 6 is true, where (3.53) follows by the uniqueness of Fréchet derivatives.

Step 7. Claim: *for every growth rate $c \in \mathscr{C}_{rd}^+\mathscr{R}(\mathbb{T},\mathbb{R})$, $a+\sigma \lhd c \unlhd b-\sigma$, the mapping $D_1(\nu_\tau,v_\tau) : \mathscr{X}\times\mathscr{P} \to \mathscr{B}_{\tau,c}^1$ is continuous.*

With a view to (3.53), it is sufficient to show the continuity of the mapping $(\nu_\tau^1,v_\tau^1) : \mathscr{X}\times\mathscr{P} \to \mathscr{B}_{\tau,c}^1$. To do this, we fix any $\xi_0 \in \mathscr{X}$, $p_0 \in \mathscr{P}$ and choose $\xi \in \mathscr{X}$, $p \in \mathscr{P}$ arbitrarily. Using the fixed point equation (3.47) for (ν_τ^1,v_τ^1), we obtain the estimate (cf. (3.48))

$$\begin{aligned}
&\left\|\begin{pmatrix}\nu_\tau^1\\ \upsilon_\tau^1\end{pmatrix}(t;\xi,p)-\begin{pmatrix}\nu_\tau^1\\ \upsilon_\tau^1\end{pmatrix}(t;\xi_0,p_0)\right\| \\
&\quad\leq \max\Bigg\{K_1\int_\tau^t e_a(t,\sigma(s)) \\
&\qquad\qquad\times\Bigg\|D_{(2,3)}F(s,(\nu_\tau,\upsilon_\tau)(s;\xi,p),p)\begin{pmatrix}\nu_\tau^1\\ \upsilon_\tau^1\end{pmatrix}(s;\xi,p) \\
&\qquad\qquad\quad - D_{(2,3)}F(s,(\nu_\tau,\upsilon_\tau)(s;\xi_0,p_0),p_0)\begin{pmatrix}\nu_\tau^1\\ \upsilon_\tau^1\end{pmatrix}(s;\xi_0,p_0)\Bigg\|\Delta s \quad \text{by (3.2)}, \\
&\qquad K_2\int_t^\infty e_b(t,\sigma(s)) \\
&\qquad\qquad\times\Bigg\|D_{(2,3)}G(s,(\nu_\tau,\upsilon_\tau)(s;\xi,p),p)\begin{pmatrix}\nu_\tau^1\\ \upsilon_\tau^1\end{pmatrix}(s;\xi,p) \\
&\qquad\qquad\quad - D_{(2,3)}G(s,(\nu_\tau,\upsilon_\tau)(s;\xi_0,p_0),p_0)\begin{pmatrix}\nu_\tau^1\\ \upsilon_\tau^1\end{pmatrix}(s;\xi_0,p_0)\Bigg\|\Delta s\Bigg\} \quad \text{for } t\in\mathbb{T}_\tau^+,
\end{aligned}\tag{3.75}$$

where subtraction and addition of the expressions

$$\begin{gathered}
D_{(2,3)}F(s,(\nu_\tau,\upsilon_\tau)(s;\xi,p),p)\begin{pmatrix}\nu_\tau^1\\ \upsilon_\tau^1\end{pmatrix}(s;\xi_0,p_0), \\
D_{(2,3)}G(s,(\nu_\tau,\upsilon_\tau)(s;\xi,p),p)\begin{pmatrix}\nu_\tau^1\\ \upsilon_\tau^1\end{pmatrix}(s;\xi_0,p_0),
\end{gathered}\tag{3.76}$$

respectively, in the corresponding norms and the use of (3.4) lead to

$$\left\|\begin{pmatrix}\nu_\tau^1\\ \upsilon_\tau^1\end{pmatrix}(t;\xi,p)-\begin{pmatrix}\nu_\tau^1\\ \upsilon_\tau^1\end{pmatrix}(t;\xi_0,p_0)\right\|\leq\max\{\alpha+\beta,\gamma+\delta\} \quad \text{for } t\in\mathbb{T}_\tau^+, \tag{3.77}$$

with the abbreviations

$$\begin{aligned}
\alpha &:= K_1\int_\tau^t e_a(t,\sigma(s))\,\|\hat{F}(s,\xi,p)\|\left\|\begin{pmatrix}\nu_\tau^1\\ \upsilon_\tau^1\end{pmatrix}(s;\xi_0,p_0)\right\|\Delta s, \\
\beta &:= K_1|F|_1\int_\tau^t e_a(t,\sigma(s))\left\|\begin{pmatrix}\nu_\tau^1\\ \upsilon_\tau^1\end{pmatrix}(s;\xi,p)-\begin{pmatrix}\nu_\tau^1\\ \upsilon_\tau^1\end{pmatrix}(s;\xi_0,p_0)\right\|\Delta s, \\
\gamma &:= K_2\int_t^\infty e_b(t,\sigma(s))\,\|\hat{G}(s,\xi,p)\|\left\|\begin{pmatrix}\nu_\tau^1\\ \upsilon_\tau^1\end{pmatrix}(s;\xi_0,p_0)\right\|\Delta s, \\
\delta &:= K_2|G|_1\int_t^\infty e_b(t,\sigma(s))\left\|\begin{pmatrix}\nu_\tau^1\\ \upsilon_\tau^1\end{pmatrix}(s;\xi,p)-\begin{pmatrix}\nu_\tau^1\\ \upsilon_\tau^1\end{pmatrix}(s;\xi_0,p_0)\right\|\Delta s,
\end{aligned}\tag{3.78}$$

$$\begin{aligned} \hat{F}(s,\xi,p) &:= D_{(2,3)}F(s,(\nu_\tau,\upsilon_\tau)(s;\xi,p),p) - D_{(2,3)}F(s,(\nu_\tau,\upsilon_\tau)(s;\xi_0,p_0),p_0), \\ \hat{G}(s,\xi,p) &:= D_{(2,3)}G(s,(\nu_\tau,\upsilon_\tau)(s;\xi,p),p) - D_{(2,3)}G(s,(\nu_\tau,\upsilon_\tau)(s;\xi_0,p_0),p_0). \end{aligned} \tag{3.79}$$

With the aid of relation (3.29), one obtains

$$\begin{aligned} &\left\| \begin{pmatrix} \nu_\tau^1 \\ \upsilon_\tau^1 \end{pmatrix}(t;\xi,p) - \begin{pmatrix} \nu_\tau^1 \\ \upsilon_\tau^1 \end{pmatrix}(t;\xi_0,p_0) \right\| e_c(\tau,t) \\ &\quad \le \alpha e_c(\tau,t) + \gamma e_c(\tau,t) + L \left\| \begin{pmatrix} \nu_\tau^1 \\ \upsilon_\tau^1 \end{pmatrix}(\xi,p) - \begin{pmatrix} \nu_\tau^1 \\ \upsilon_\tau^1 \end{pmatrix}(\xi_0,p_0) \right\|_{\tau,c}^{+} \quad \text{for } t \in \mathbb{T}_\tau^+ \quad \text{by (3.24).} \end{aligned} \tag{3.80}$$

We define $c_1 := a + \sigma$ to get $(\nu_\tau^1, \upsilon_\tau^1)(\xi_0, p_0) \in \mathscr{B}_{\tau,c_1}^1$. In the integrals α and γ, we can estimate the mapping $(\nu_\tau^1, \upsilon_\tau^1)(\xi_0, p_0)$ using its c_1^+-norm, which yields

$$\begin{aligned} \alpha &\le K_1 \left\| \begin{pmatrix} \nu_\tau^1 \\ \upsilon_\tau^1 \end{pmatrix}(\xi_0,p_0) \right\|_{\tau,c_1}^{+} \int_\tau^t e_a(t,\sigma(s)) e_{c_1}(s,\tau) \|\hat{F}(s,\xi,p)\| \Delta s \quad \text{for } t \in \mathbb{T}_\tau^+, \\ \gamma &\le K_2 \left\| \begin{pmatrix} \nu_\tau^1 \\ \upsilon_\tau^1 \end{pmatrix}(\xi_0,p_0) \right\|_{\tau,c_1}^{+} \int_t^\infty e_b(t,\sigma(s)) e_{c_1}(s,\tau) \|\hat{G}(s,\xi,p)\| \Delta s \quad \text{for } t \in \mathbb{T}_\tau^+. \end{aligned} \tag{3.81}$$

Now we substitute these expressions into (3.80) and pass over to the supremum over $t \in \mathbb{T}_\tau^+$ to derive

$$\left\| \begin{pmatrix} \nu_\tau^1 \\ \upsilon_\tau^1 \end{pmatrix}(\xi,p) - \begin{pmatrix} \nu_\tau^1 \\ \upsilon_\tau^1 \end{pmatrix}(\xi_0,p_0) \right\|_{\tau,c}^{+} \le \frac{\max\{K_1,K_2\}}{1-L} \left\| \begin{pmatrix} \nu_\tau^1 \\ \upsilon_\tau^1 \end{pmatrix}(\xi_0,p_0) \right\|_{\tau,c_1}^{+} \sup_{\tau \preceq t} W(t,\xi,p) \tag{3.82}$$

by (3.24) with

$$\begin{aligned} W(t,\xi,p) := &\int_\tau^t e_a(t,\sigma(s)) e_{c_1}(s,\tau) \|\hat{F}(s,\xi,p)\| \Delta s \\ &+ \int_t^\infty e_b(t,\sigma(s)) e_{c_1}(s,\tau) \|\hat{G}(s,\xi,p)\| \Delta s. \end{aligned} \tag{3.83}$$

Therefore it is sufficient to prove the limit relation

$$\lim_{(\xi,p)\to(\xi_0,p_0)} \sup_{\tau \preceq t} W(t,\xi,p) = 0 \tag{3.84}$$

to show the claim in Step 7. We proceed indirectly and assume (3.84) does not hold. Then there exist an $\epsilon > 0$ and a sequence $((\xi_i, p_i))_{i\in\mathbb{N}}$ in $\mathscr{X} \times \mathscr{P}$ with $\lim_{i\to\infty}(\xi_i, p_i) = (\xi_0, p_0)$ and

$$\sup_{\tau \preceq t} W(t,\xi_i,p_i) > \epsilon \quad \text{for } i \in \mathbb{N}, \tag{3.85}$$

which moreover leads to the existence of a sequence $(t_i)_{i\in\mathbb{N}}$ in $\mathbb{T}_\tau^+$ such that

$$W(t_i,\xi_i,p_i) > \epsilon \quad \text{for } i \in \mathbb{N}. \tag{3.86}$$

Apart from this, we get (cf. (3.4), (3.79))

$$\begin{aligned} W(t,\xi,p) &\le 2|F|_1\int_\tau^t e_a(t,\sigma(s))e_{c_1}(s,\tau)\Delta s+2|G|_1\int_t^\infty e_b(t,\sigma(s))e_{c_1}(s,\tau)\Delta s \quad \text{by (3.83)} \\ &\le \left(\frac{2|F|_1}{\lfloor c_1-a\rfloor}+\frac{2|G|_1}{\lfloor b-c_1\rfloor}\right)e_{c_1\ominus c}(t,\tau) \end{aligned} \tag{3.87}$$

for $t\in\mathbb{T}_\tau^+$, and since $c_1\lhd c$, the right-hand side of this estimate converges to 0 for $t\to\infty$, which yields $\lim_{t\to\infty}W(t,\xi,p)=0$ uniformly in $(\xi,p)\in\mathscr{X}\times\mathscr{P}$. Because of (3.86), the sequence $(t_i)_{i\in\mathbb{N}}$ in $\mathbb{T}_\tau^+$ has to be bounded above, that is, there exists a time $T\in(\tau,\infty)_\mathbb{T}$ with $t_i\preceq T$ for all $i\in\mathbb{N}$, and this is used to obtain

$$\begin{aligned} W(t_i,\xi_i,p_i) \le &\int_\tau^T e_c(\tau,\sigma(s))e_{c_1}(s,\tau)\|\hat F(s,\xi,p)\|\Delta s \\ &+\int_\tau^\infty e_c(\tau,\sigma(s))e_{b\ominus c}(T,\sigma(s))e_{c_1}(s,\tau)\|\hat G(s,\xi,p)\|\Delta s \quad \text{for } i\in\mathbb{N}. \end{aligned} \tag{3.88}$$

The continuity of $(\nu_\tau,\upsilon_\tau)(s,\cdot)$ from Step 1 gives us the relation

$$\lim_{i\to\infty}\begin{pmatrix}\nu_\tau\\ \upsilon_\tau\end{pmatrix}(s;\xi_i,p_i)=\begin{pmatrix}\nu_\tau\\ \upsilon_\tau\end{pmatrix}(s;\xi_0,p_0) \quad \text{for } s\in\mathbb{T}_\tau^+, \tag{3.89}$$

and therefore the finite integral in (3.88) tends to 0 for $i\to\infty$ by (3.79) and the continuity of $D_{(2,3)}F$. By the continuity of $D_{(2,3)}G$, the indefinite integral in (3.88) does the same, and we can apply Lebesgue's theorem, which finally implies $\lim_{i\to\infty}W(t_i,\xi_i,p_i)=0$. Of course this contradicts (3.86), and consequently we have shown the above claim in Step 7.

Step 8. We have the identity $s(\tau,\xi,p)=\upsilon_\tau(\xi,p)(\tau)$ for $\tau\in\mathbb{T}$, $\xi\in\mathscr{X}$, $p\in\mathscr{P}$, and by well-known properties of the evaluation map (see [18, Lemma 3.4]), it follows that the mapping $s(\tau,\cdot):\mathscr{X}\times\mathscr{P}\to\mathscr{Y}$, $\tau\in\mathbb{T}$, is continuously differentiable with respect to its variable in $\mathscr{X}$. We do not show that $D_2s:\mathbb{T}\times\mathscr{X}\times\mathscr{P}\to\mathscr{Y}$ is continuous here. This can be seen by carrying over arguments developed for ordinary differential equations in [24, pages 160–163] to dynamic equations (cf. [17, Lemma 3.1.3(a), page 130]). Thereto one has to assume that the parameter space $\mathscr{P}$ is locally compact. Finally, the existence and rd-continuity of $\Delta_1 s:\mathbb{T}\times\mathscr{X}\times\mathscr{P}\to\mathscr{Y}$ result from [17, Lemma 3.1.3(b), page 130] together with the continuity of D_2s.

(b) Since part (b) of the theorem can be proved along the same lines of part (a), we present only a rough sketch of the proof. Analogously to Lemma 3.4, for initial values $\eta\in\mathscr{Y}$ and parameters $p\in\mathscr{P}$, the c^--quasibounded solutions of system (3.1) may be characterized as the fixed points of a mapping $\bar{\mathscr{T}}_\tau:\mathscr{B}_{\tau,c}^-(\mathscr{X}\times\mathscr{Y})\times\mathscr{Y}\times\mathscr{P}\to\mathscr{B}_{\tau,c}^-(\mathscr{X}\times\mathscr{Y})$,

$$\bar{\mathscr{T}}_\tau(\nu,\upsilon;\eta,p):=\begin{pmatrix}\displaystyle\int_{-\infty}^{\cdot}\Phi_A(\cdot,\sigma(s))F(s,(\nu,\upsilon)(s),p)\Delta s\\ \displaystyle\Phi_B(\cdot,\tau)\eta+\int_\tau^t\Phi_B(\cdot,\sigma(s))G(s,(\nu,\upsilon)(s),p)\Delta s\end{pmatrix}. \tag{3.90}$$

Now, $\bar{\mathcal{T}}_\tau$ can be treated just as $\mathcal{T}_\tau$ in (a). In order to prove the counterpart of Lemma 3.3, the two results [19, Theorems 2(a) and 4(b)] have to be replaced by [19, Theorems 4(a) and 2(b)]. It follows from assumption (3.5) that also $\bar{\mathcal{T}}_\tau$ is a contraction on $\mathcal{B}^-_{\tau,c}(\mathcal{X}\times\mathcal{Y})$ and if $(\nu_\tau,\upsilon_\tau)(\eta,p)\in\mathcal{B}^-_{\tau,c}(\mathcal{X}\times\mathcal{Y})$ denotes its unique fixed point, we define the function $r:\mathbb{T}\times\mathcal{Y}\times\mathcal{P}\to\mathcal{X}$ by $r(\tau,\eta,p):=(\nu_\tau(\eta,p))(\tau)$. The claimed properties of r can be proved along the lines of part (a).

(c) The proof of part (c) has been carried out in [19, Theorem 4.9(c)] and we have established the proof of Theorem 3.5 completely. □

4. Higher-order smoothness of invariant fiber bundles

In [18] we proved a higher-order smoothness result for the fiber bundle S or R in only a nearly hyperbolic situation, that is, if the growth rates a, b and the real σ from Hypothesis 3.1 satisfy $a+\sigma\trianglelefteq 0$ or $0\trianglelefteq b-\sigma$, respectively. Now we weaken this assumption and replace it by the so-called gap condition. This, however, needs some technical preparations.

Lemma 4.1. *Assume $m\in\mathbb{N}$ and that $a,b\in\mathscr{C}^+_{rd}\mathcal{R}(\mathbb{T},\mathbb{R})$ are growth rates.*

(a) *Under the gap condition $m\odot a\lhd b$, the mapping $\rho_s^m[a,b]:\mathbb{T}\to\mathbb{R}$,*

$$\rho_s^m[a,b](t):=\lim_{h\searrow\mu^*(t)}\frac{1+ha(t)}{h}\left(\sqrt[m]{\frac{1+ha(t)+1+hb(t)}{1+ha(t)+(1+ha(t))^m}}-1\right),\tag{4.1}$$

satisfies $\lfloor\rho_s^m[a,b]\rfloor>0$.

(b) *Under the gap condition $a\lhd m\odot b$, the mapping $\rho_r^m[a,b]:\mathbb{T}\to\mathbb{R}$,*

$$\rho_r^m[a,b](t):=\lim_{h\searrow\mu^*(t)}\frac{1+hb(t)}{h}\left(1-\sqrt[m]{\frac{1+ha(t)+1+hb(t)}{1+hb(t)+(1+hb(t))^m}}\right),\tag{4.2}$$

satisfies $\lfloor\rho_r^m[a,b]\rfloor>0$.

Proof. We establish only (a) since statement (b) follows analogously. In the proof, one has to distinguish the cases $\mu^*(t)=0$, where l'Hospital's rule yields

$$\lim_{h\searrow 0}\frac{1+ha(t)}{h}\left(\sqrt[m]{\frac{1+ha(t)+1+hb(t)}{1+ha(t)+(1+ha(t))^m}}-1\right)=\frac{b(t)-ma(t)}{2m},\tag{4.3}$$

and $\mu^*(t)>0$, where the assertion follows by easy estimates from the condition $m\odot a\lhd b$ since a,b are growth rates and since μ^* is bounded above. □

This leads to the main result of this paper.

Theorem 4.2 ($\mathscr{C}^m$-smoothness). *Assume Hypothesis 3.1. Then the assertions of Theorem 3.5 hold and moreover the mappings s and r satisfy the following statements.*

(a) *Under the gap condition*

$$m_s \odot a \lhd b \tag{4.4}$$

for $m_s \in \{1,\dots,m\}$ *and if* $\sigma_{\max} = \min\{\lfloor b-a\rfloor/2, \lfloor \rho_s^m[a,b]\rfloor\}$, *the mapping* $s(\tau,\cdot) : \mathscr{X}\times\mathscr{P}\to\mathscr{Y}$, $\tau\in\mathbb{T}$, *is* m_s*-times continuously differentiable in the argument* $\xi\in\mathscr{X}$ *with globally bounded derivatives*

$$\|D_2^n s(\tau,\xi,p)\|_{\mathscr{L}_n(\mathscr{X};\mathscr{Y})} \le C_n \quad \text{for } n\in\{1,\dots,m_s\},\ (\tau,\xi,p)\in\mathbb{T}\times\mathscr{X}\times\mathscr{P}, \tag{4.5}$$

where in particular $C_1 := \sigma K_1/(\sigma - \max\{K_1|F|_1, K_2|G|_1\})$.

(b) *In case* $\mathbb{T}$ *is unbounded below, under the gap condition*

$$a \lhd m_r \odot b \tag{4.6}$$

for $m_r \in \{1,\dots,m\}$ *and if* $\sigma_{\max} = \min\{\lfloor b-a\rfloor/2, \lfloor \rho_s^m[a,b]\rfloor\}$, *the mapping* $r(\tau,\cdot) : \mathscr{Y}\times\mathscr{P}\to\mathscr{X}$, $\tau\in\mathbb{T}$, *is* m_r*-times continuously differentiable in the argument* $\eta\in\mathscr{Y}$ *with globally bounded derivatives*

$$\|D_2^n r(\tau,\eta,p)\|_{\mathscr{L}_n(\mathscr{Y};\mathscr{X})} \le C_n \quad \text{for } n\in\{1,\dots,m_r\},\ (\tau,\eta,p)\in\mathbb{T}\times\mathscr{Y}\times\mathscr{P}, \tag{4.7}$$

where in particular $C_1 := \sigma K_2/(\sigma - \max\{K_1|F|_1, K_2|G|_1\})$.

(c) *The global bounds* $C_2,\dots,C_m \ge 0$ *can be determined recursively using the formula*

$$C_n := \frac{\max\left\{K_1\sum_{j=2}^n |F|_j \sum_{(N_1,\dots,N_j)\in P_j^<(n)} \prod_{i=1}^j C_{\#N_i}, K_2\sum_{j=2}^n |G|_j \sum_{(N_1,\dots,N_j)\in P_j^<(n)} \prod_{i=1}^j C_{\#N_i}\right\}}{\sigma - \max\{K_1,K_2\}\max\{|F|_1,|G|_1\}} \tag{4.8}$$

for $n\in\{2,\dots,m\}$.

Remark 4.3. In the case of constant growth rates and homogeneous measure chains, that is, for ordinary differential equations and ordinary difference equations, the above gap condition (4.4) is sharp, that is, for example, the invariant fiber bundle S from Theorem 3.5(a) is only of class $\mathscr{C}^{m_s}$ in general, even if the nonlinearities F and G are $\mathscr{C}^\infty$-functions. This is demonstrated in [20, Example 5.2] for difference equations.

Proof. (a) Since the proof is quite involved, we subdivide it into six steps and use the conventions and notation from the proof of Theorem 3.5 for brevity. We choose $\tau\in\mathbb{T}$.

Step I. Let $c \in \mathscr{C}^+_{\mathrm{rd}}\mathscr{R}(\mathbb{T},\mathbb{R})$, $a+\sigma \trianglelefteq c \trianglelefteq b-\sigma$, and let $\xi \in \mathscr{X}$, $p \in \mathscr{P}$ be arbitrary. By formal differentiation of the fixed point equation (3.46) with respect to $\xi \in \mathscr{X}$, using the higher-order chain rule from Theorem 2.1, we obtain another fixed point equation

$$\begin{pmatrix} \nu^l_\tau \\ \upsilon^l_\tau \end{pmatrix}(\xi,p) = \mathscr{T}^l_\tau\big((\nu^l_\tau,\upsilon^l_\tau)(\xi,p);\xi,p\big) \tag{4.9}$$

for the formal partial derivative $(\nu^l_\tau,\upsilon^l_\tau)$ of $(\nu_\tau,\upsilon_\tau) : \mathscr{X}\times\mathscr{P} \to \mathscr{B}^+_{\tau,c}(\mathscr{X}\times\mathscr{Y})$ of order $l \in \{2,\dots,m_s\}$, where the right-hand side of (4.9) is given by

$$\begin{aligned}
&\mathscr{T}^l_\tau(\nu^l,\upsilon^l;\xi,p)\\
&\quad:= \begin{pmatrix} \displaystyle\int_\tau^\cdot \Phi_A(\cdot,\sigma(s))\left[D_{(2,3)}F(s,(\nu_\tau,\upsilon_\tau)(s;\xi,p),p)\begin{pmatrix}\nu^l\\ \upsilon^l\end{pmatrix}(s) + R^l_1(s,\xi,p)\right]\Delta s \\ \displaystyle -\int_\cdot^\infty \Phi_B(\cdot,\sigma(s))\left[D_{(2,3)}G(s,(\nu_\tau,\upsilon_\tau)(s;\xi,p),p)\begin{pmatrix}\nu^l\\ \upsilon^l\end{pmatrix}(s) + R^l_2(s,\xi,p)\right]\Delta s \end{pmatrix}.
\end{aligned} \tag{4.10}$$

Here, (ν^l,υ^l) is a mapping from $\mathbb{T}^+_\tau$ to $\mathscr{L}_l(\mathscr{X};\mathscr{X}\times\mathscr{Y})$. The remainder $R^l = (R^l_1,R^l_2)$ has the following two representations:

(1) as a partially unfolded derivative tree,

$$R^l(s,\xi,p) = \sum_{j=1}^{l-1}\binom{l-1}{j}\frac{\partial^j}{\partial\xi^j}\big[D_{(2,3)}(F,G)(s,(\nu_\tau,\upsilon_\tau)(s;\xi,p),p)\big]\begin{pmatrix}\nu^{l-j}_\tau\\ \upsilon^{l-j}_\tau\end{pmatrix}(s;\xi,p) \quad \text{by (2.3)}, \tag{4.11}$$

which is appropriate for the induction in the subsequent step (Step IV),

(2) as a totally unfolded derivative tree,

$$\begin{aligned} R^l(s,\xi,p) = \sum_{j=2}^{l}\sum_{(N_1,\dots,N_j)\in P^<_j(l)} & D^j_{(2,3)}(F,G)(s,(\nu_\tau,\upsilon_\tau)(s;\xi,p),p)\\ &\times\begin{pmatrix}\nu^{\#N_1}_\tau\\ \upsilon^{\#N_1}_\tau\end{pmatrix}(s;\xi,p)\cdots\begin{pmatrix}\nu^{\#N_j}_\tau\\ \upsilon^{\#N_j}_\tau\end{pmatrix}(s;\xi,p) \quad \text{by (2.4)}, \end{aligned} \tag{4.12}$$

which enables us to obtain explicit global bounds for the higher-order derivatives in Step II. For our forthcoming considerations, it is crucial that R^l does not depend on $(\nu^l_\tau,\upsilon^l_\tau)$. In the following steps, we will solve the fixed point equation (4.9) for the operator $\mathscr{T}^l_\tau$. As a preparation, we introduce for every $l \in \{1,\dots,m_s\}$ the abbreviations $c_l := \max\{a+\sigma, l\odot(a+\sigma)\}$; it is

$$c_l(t) = \begin{cases} a(t)+\sigma, & \text{if } a(t)+\sigma \le 0,\\ (l\odot(a+\sigma))(t), & \text{if } 0 \le a(t)+\sigma, \end{cases} \quad \text{for } t \in \mathbb{T}. \tag{4.13}$$

Then $c_1,\dots,c_{m_s}$ are growth rates because of the gap condition (4.8) and with our choice of $\sigma_{\max}$, it is easy to see that one has the inequality $a+\sigma \trianglelefteq c_1,\dots,c_{m_s} \triangleleft b-\sigma$, which in case $a(t)+\sigma \le 0$ follows from $\sigma < \lfloor b-a \rfloor /2$ and otherwise essentially results from $m_s \odot (a+\sigma) \lhd b-\sigma$, which in turn is implied by

$$\begin{aligned}
&(1+h(a(t)+\sigma)^m)+1+h(a(t)+\sigma)\\
&\quad = (1+ha(t))^m\left(1+\frac{h\sigma}{1+ha(t)}\right)^m+(1+ha(t))\left(1+\frac{h\sigma}{1+ha(t)}\right)\\
&\quad \le \left[(1+ha(t))^m+1+ha(t)\right]\left(1+\frac{h\sigma}{1+h\sigma}\right)^m\\
&\quad < 1+ha(t)+1+hb(t) \quad \text{for } t \in \mathbb{T},
\end{aligned} \tag{4.14}$$

if $\sigma < \lfloor \rho_s^m[a,b] \rfloor$ (cf. Lemma 4.1). Now we formulate for $\bar{m} \in \{1,\dots,m_s\}$ the induction hypotheses.

$A(\bar{m})$ For any $l \in \{1,\dots,\bar{m}\}$ and growth rates $c \in \mathscr{C}_{\mathrm{rd}}^+\mathscr{R}(\mathbb{T},\mathbb{R}), c_l \preceq c \prec b-\sigma$, the operator $\mathscr{T}_\tau^l : \mathscr{B}_{\tau,c}^l \times \mathscr{X} \times \mathscr{P} \to \mathscr{B}_{\tau,c}^l$ satisfies the following:

(a) it is well defined,

(b) $\mathscr{T}_\tau^l(\cdot;\xi,p)$ is a uniform contraction in $\xi \in \mathscr{X}$, $p \in \mathscr{P}$,

(c) the unique fixed point $(\nu_\tau^l,\upsilon_\tau^l)(\cdot;\xi,p) = (\nu_\tau^l,\upsilon_\tau^l)(\xi,p)$ of $\mathscr{T}_\tau^l(\cdot;\xi,p)$ is globally bounded in the c_l^+-norm

$$\left\| \begin{pmatrix} \nu_\tau^l \\ \upsilon_\tau^l \end{pmatrix}(s;\xi,p) \right\| \le C_l e_{c_l}(s,\tau) \quad \text{for } s \in \mathbb{T}_\tau^+,\ \xi \in \mathscr{X},\ p \in \mathscr{P}, \tag{4.15}$$

with the constants $C_l \ge 0$ given in (4.8),

(d) if $c_l \lhd c$, then $(\nu_\tau^{l-1},\upsilon_\tau^{l-1}) : \mathscr{X} \times \mathscr{P} \to \mathscr{B}_{\tau,c}^l$ is continuously partially differentiable with respect to $\xi \in \mathscr{X}$ with derivative

$$D_1 \begin{pmatrix} \nu_\tau^{l-1} \\ \upsilon_\tau^{l-1} \end{pmatrix} = \begin{pmatrix} \nu_\tau^l \\ \upsilon_\tau^l \end{pmatrix} : \mathscr{X} \times \mathscr{P} \to \mathscr{B}_{\tau,c}^l. \tag{4.16}$$

For $\bar{m}=1$, the proof of Theorem 3.5 implies the induction hypothesis $A(1)$ with $C_1 = K_1/(1-L)$ (cf. (3.51)). Now we are assuming that $A(\bar{m}-1)$ holds true for an $\bar{m} \in \{2,\dots,m_s\}$ and we are going to prove $A(\bar{m})$ in the following five steps.

Step II. Claim: *for every growth rate $c \in \mathscr{C}_{rd}^+\mathscr{R}(\mathbb{T},\mathbb{R})$, $c_{\bar{m}} \unlhd c \lhd b-\sigma$, the operator $\mathscr{T}_\tau^{\bar{m}} : \mathscr{B}_{\tau,c}^{\bar{m}} \times \mathscr{X} \times \mathscr{P} \to \mathscr{B}_{\tau,c}^{\bar{m}}$ is well defined and satisfies the estimate*

$$\begin{aligned}
&\|\mathscr{T}_\tau^{\bar{m}}(\nu^{\bar{m}},\upsilon^{\bar{m}};\xi,p)\|_{\tau,c}^+\\
&\le L\left\|\begin{pmatrix} \nu^{\bar{m}} \\ \upsilon^{\bar{m}} \end{pmatrix}\right\|_{\tau,c}^+ + \max\left\{\frac{K_1}{\sigma}\sum_{j=2}^{\bar{m}}|F|_j \sum_{(N_1,\dots,N_j)\in P_j^<(\bar{m})}\prod_{i=1}^j C_{\#N_i}, \frac{K_2}{\sigma}\sum_{j=2}^{\bar{m}}|G|_j \sum_{(N_1,\dots,N_j)\in P_j^<(\bar{m})}\prod_{i=1}^j C_{\#N_i}\right\}\\
&\qquad\qquad \text{for } (\nu^{\bar{m}},\upsilon^{\bar{m}}) \in \mathscr{B}_{\tau,c}^{\bar{m}},\ \xi \in \mathscr{X},\ p \in \mathscr{P},
\end{aligned} \tag{4.17}$$

that is, $A(\bar{m})(a)$ holds.

Let $l \in \{2,\dots,\bar{m}\}$, $\xi \in \mathscr{X}$, $p \in \mathscr{P}$ be arbitrary and choose $c \in \mathscr{C}^+_{\mathrm{rd}}\mathscr{R}(\mathbb{T},\mathbb{R})$, $c_l \trianglelefteq c \lhd b - \sigma$. Using the estimate $c_{\#N_1} \oplus \cdots \oplus c_{\#N_j} \trianglelefteq c_l$ for any ordered partition $(N_1,\dots,N_j) \in P_j^<(l)$ of length $j \in \{2,\dots,l\}$, from (3.2), (3.4), and $A(\bar{m}-1)(c)$, we obtain the inequality

$$
\begin{aligned}
&\left\|\begin{pmatrix} \displaystyle\int_\tau^t \Phi_A(t,\sigma(s))R_1^l(s,\xi,p)\Delta s \\ \displaystyle -\int_t^\infty \Phi_B(t,\sigma(s))R_2^l(s,\xi,p)\Delta s \end{pmatrix}\right\| \\
&\quad \le \max\left\{K_1\int_\tau^t e_a(t,\sigma(s))\sum_{j=2}^{l}|F|_j \sum_{(N_1,\dots,N_j)\in P_j^<(l)}\prod_{i=1}^{j} C_{\#N_i}e_{c_{\#N_i}}(s,\tau)\Delta s,\right. \\
&\qquad\quad \left. K_2\int_t^\infty e_b(t,\sigma(s))\sum_{j=2}^{l}|G|_j \sum_{(N_1,\dots,N_j)\in P_j^<(l)}\prod_{i=1}^{j} C_{\#N_i}e_{c_{\#N_i}}(s,\tau)\Delta s\right\} \quad \text{by (4.12)} \\
&\quad \le \max\left\{K_1\int_\tau^t e_a(t,\sigma(s))e_{c_l}(s,\tau)\sum_{j=2}^{l}|F|_j \sum_{(N_1,\dots,N_j)\in P_j^<(l)}\prod_{i=1}^{j} C_{\#N_i}\Delta s,\right. \\
&\qquad\quad \left. K_2\int_t^\infty e_b(t,\sigma(s))e_{c_l}(s,\tau)\sum_{j=2}^{l}|G|_j \sum_{(N_1,\dots,N_j)\in P_j^<(l)}\prod_{i=1}^{j} C_{\#N_i}\Delta s\right\} \\
&\quad \le \max\left\{\frac{K_1}{\lfloor c_l - a\rfloor}\sum_{j=2}^{l}|F|_j \sum_{(N_1,\dots,N_j)\in P_j^<(l)}\prod_{i=1}^{j} C_{\#N_i}, \frac{K_2}{\lfloor b - c_l\rfloor}\right. \\
&\qquad\quad \left. \times\sum_{j=2}^{l}|G|_j \sum_{(N_1,\dots,N_j)\in P_j^<(l)}\prod_{i=1}^{j} C_{\#N_i}\right\} e_c(t,\tau)
\end{aligned}
\tag{4.18}
$$

for $t \in \mathbb{T}_\tau^+$ by [8, Theorem 7.4(i)]. Now, let $c \in \mathscr{C}^+_{\mathrm{rd}}\mathscr{R}(\mathbb{T},\mathbb{R})$, $c_{\bar{m}} \trianglelefteq c \lhd b - \sigma$, be arbitrary but fixed, and $(\nu^{\bar{m}}, \upsilon^{\bar{m}}) \in \mathscr{B}^{\bar{m}}_{\tau,c}$. With the aid of the above estimate (4.18), we obtain

$$
\begin{aligned}
&\|\mathscr{T}_\tau^{\bar{m}}(\nu^{\bar{m}},\upsilon^{\bar{m}};\xi,p)(t)\| \\
&\quad \le \max\left\{K_1|F|_1\int_\tau^t e_a(t,\sigma(s))e_c(s,\tau)\Delta s\left\|\begin{pmatrix}\nu^{\bar{m}}\\ \upsilon^{\bar{m}}\end{pmatrix}\right\|^+_{\tau,c}\right. \\
&\qquad\quad + \frac{K_1}{\lfloor c_{\bar{m}} - a\rfloor}\sum_{j=2}^{\bar{m}}|F|_j \sum_{(N_1,\dots,N_j)\in P_j^<(\bar{m})}\prod_{i=1}^{j} C_{\#N_i}e_c(t,\tau), \\
&\qquad\quad K_2|G|_1\int_t^\infty e_b(t,\sigma(s))e_c(s,\tau)\Delta s\left\|\begin{pmatrix}\nu^{\bar{m}}\\ \upsilon^{\bar{m}}\end{pmatrix}\right\|^+_{\tau,c} \\
&\qquad\quad \left. + \frac{K_2}{\lfloor b - c_{\bar{m}}\rfloor}\sum_{j=2}^{\bar{m}}|G|_j \sum_{(N_1,\dots,N_j)\in P_j^<(\bar{m})}\prod_{i=1}^{j} C_{\#N_i}e_c(t,\tau)\right\} \quad \text{by (4.10)}
\end{aligned}
$$

$$\leq \max\left\{\frac{K_1|F|_1}{\lfloor c-a\rfloor}\left\|\begin{pmatrix}\nu^{\bar m}\\ \upsilon^{\bar m}\end{pmatrix}\right\|_{\tau,c}^{+}+\frac{K_1}{\lfloor c_{\bar m}-a\rfloor}\sum_{j=2}^{\bar m}|F|_j\sum_{(N_1,\dots,N_j)\in P_j^<(\bar m)}\prod_{i=1}^{j}C_{\#N_i},\frac{K_2|G|_1}{\lfloor b-c\rfloor}\left\|\begin{pmatrix}\nu^{\bar m}\\ \upsilon^{\bar m}\end{pmatrix}\right\|_{\tau,c}^{+}\right.$$
$$\left.+\frac{K_2}{\lfloor b-c_{\bar m}\rfloor}\sum_{j=2}^{\bar m}|G|_j\sum_{(N_1,\dots,N_j)\in P_j^<(\bar m)}\prod_{i=1}^{j}C_{\#N_i}\right\}e_c(t,\tau)$$
$$\leq L\left\|\begin{pmatrix}\nu^{\bar m}\\ \upsilon^{\bar m}\end{pmatrix}\right\|_{\tau,c}^{+}$$
$$+\max\left\{\frac{K_1}{\lfloor c_{\bar m}-a\rfloor}\sum_{j=2}^{\bar m}|F|_j\sum_{(N_1,\dots,N_j)\in P_j^<(\bar m)}\prod_{i=1}^{j}C_{\#N_i},\right.$$
$$\left.\frac{K_2}{\lfloor b-c_{\bar m}\rfloor}\sum_{j=2}^{\bar m}|G|_j\sum_{(N_1,\dots,N_j)\in P_j^<(\bar m)}\prod_{i=1}^{j}C_{\#N_i}\right\}e_c(t,\tau)\quad\text{for } t\in\mathbb{T}_\tau^+\quad\text{by (3.24)},\tag{4.19}$$

and after multiplying this by $e_c(\tau,t)$, passing over to the least upper bound over $t\in\mathbb{T}_\tau^+$ implies our claim $\mathcal{T}_\tau^{\bar m}(\nu^{\bar m},\upsilon^{\bar m};\xi,p)\in\mathcal{B}_{\tau,c}^{\bar m}$. In particular, the estimate (4.17) is a consequence of (4.19) and the choice of $a+\sigma\trianglelefteq c_{\bar m}\triangleleft b-\sigma$.

Step III. Claim: *for every growth rate $c\in\mathcal{C}_{rd}^+\mathcal{R}(\mathbb{T},\mathbb{R})$, $c_{\bar m}\trianglelefteq c\triangleleft b-\sigma$, the operator $\mathcal{T}_\tau^{\bar m}(\cdot;\xi,p):\mathcal{B}_{\tau,c}^{\bar m}\to\mathcal{B}_{\tau,c}^{\bar m}$ is a uniform contraction in $\xi\in\mathcal{X}$, $p\in\mathcal{P}$; moreover, the fixed point $(\nu_\tau^{\bar m},\upsilon_\tau^{\bar m})(\xi,p)\in\mathcal{B}_{\tau,c}^{\bar m}$ does not depend on $c\in\mathcal{C}_{rd}^+\mathcal{R}(\mathbb{T},\mathbb{R})$, $c_{\bar m}\trianglelefteq c\triangleleft b-\sigma$, and satisfies*

$$\left\|\begin{pmatrix}\nu_\tau^{\bar m}\\ \upsilon_\tau^{\bar m}\end{pmatrix}(\xi,p)\right\|_{\tau,c}^{+}\leq C_{\bar m}\quad\text{for }\xi\in\mathcal{X},\ p\in\mathcal{P},\tag{4.20}$$

that is, $A(\bar m)(b)$ and (c) hold.

Choose $c\in\mathcal{C}_{\mathrm{rd}}^+\mathcal{R}(\mathbb{T},\mathbb{R})$, $c_{\bar m}\trianglelefteq c\triangleleft b-\sigma$, arbitrarily but fixed, and let $(\nu^{\bar m},\upsilon^{\bar m}),(\bar\nu^{\bar m},\bar\upsilon^{\bar m})\in\mathcal{B}_{\tau,c}^{\bar m}$, $\xi\in\mathcal{X}$, $p\in\mathcal{P}$. Keeping in mind that the remainder $R^{\bar m}$ does not depend on $(\nu^{\bar m},\upsilon^{\bar m})$ or $(\bar\nu^{\bar m},\bar\upsilon^{\bar m})$, respectively, from (3.2) and (3.4), we obtain the Lipschitz estimate

$$\|\mathcal{T}_\tau^{\bar m}(\nu^{\bar m},\upsilon^{\bar m};\xi,p)(t)-\mathcal{T}_\tau^{\bar m}(\bar\nu^{\bar m},\bar\upsilon^{\bar m};\xi,p)(t)\|e_c(\tau,t)$$
$$\leq\max\left\{K_1|F|_1\int_\tau^t e_a(t,\sigma(s))\left\|\begin{pmatrix}\nu^{\bar m}\\ \upsilon^{\bar m}\end{pmatrix}(s)-\begin{pmatrix}\bar\nu^{\bar m}\\ \bar\upsilon^{\bar m}\end{pmatrix}(s)\right\|\Delta s,\right.$$
$$\left.K_2|G|_1\int_t^\infty e_b(t,\sigma(s))\left\|\begin{pmatrix}\nu^{\bar m}\\ \upsilon^{\bar m}\end{pmatrix}(s)-\begin{pmatrix}\bar\nu^{\bar m}\\ \bar\upsilon^{\bar m}\end{pmatrix}(s)\right\|\Delta s\right\}e_c(\tau,t)\quad\text{by (4.10)}$$
$$\leq\max\left\{K_1|F|_1\int_\tau^t e_a(t,\sigma(s))e_c(s,\tau)\Delta s,K_2|G|_1\int_t^\infty e_b(t,\sigma(s))e_c(s,\tau)\Delta s\right\}$$
$$\cdot\, e_c(\tau,t)\left\|\begin{pmatrix}\nu^{\bar m}\\ \upsilon^{\bar m}\end{pmatrix}-\begin{pmatrix}\bar\nu^{\bar m}\\ \bar\upsilon^{\bar m}\end{pmatrix}\right\|_{\tau,c}^{+}$$

$$\begin{aligned}
&\leq \max\left\{\frac{K_1|F|_1}{\lfloor c-a\rfloor},\frac{K_2|G|_1}{\lfloor b-c\rfloor}\right\}\left\|\begin{pmatrix}\nu^{\bar m}\\ \upsilon^{\bar m}\end{pmatrix}-\begin{pmatrix}\bar\nu^{\bar m}\\ \bar\upsilon^{\bar m}\end{pmatrix}\right\|^+_{\tau,c}\\
&\leq L\left\|\begin{pmatrix}\nu^{\bar m}\\ \upsilon^{\bar m}\end{pmatrix}-\begin{pmatrix}\bar\nu^{\bar m}\\ \bar\upsilon^{\bar m}\end{pmatrix}\right\|^+_{\tau,c}\quad \text{for } t\in\mathbb{T}^+_\tau \quad \text{by (3.24)},
\end{aligned} \tag{4.21}$$

and passing over to the least upper bound over $t\in\mathbb{T}^+_\tau$ together with (3.15) implies our claim. Therefore Banach's fixed point theorem guarantees the unique existence of a fixed point $(\nu^{\bar m}_\tau,\upsilon^{\bar m}_\tau)(\xi,p)\in\mathscr{B}^{\bar m}_{\tau,c}$ of the mapping $\mathcal{T}^{\bar m}_\tau(\cdot;\xi,p):\mathscr{B}^{\bar m}_{\tau,c}\to\mathscr{B}^{\bar m}_{\tau,c}$. It can be seen along the same lines as in Step 5 in the proof of Theorem 3.5 that $(\nu^{\bar m}_\tau,\upsilon^{\bar m}_\tau)(\xi,p)$ does not depend on $c\in\mathscr{C}^+_{\mathrm{rd}}\mathscr{R}(\mathbb{T},\mathbb{R})$, $c_{\bar m}\trianglelefteq c\triangleleft b-\sigma$. The fixed point identity (4.9) for $(\nu^{\bar m}_\tau,\upsilon^{\bar m}_\tau)(\xi,p)$, together with (4.17) and (3.15), finally implies (4.20).

Step IV. Claim: *for every growth rate $c\in\mathscr{C}^+_{\mathrm{rd}}\mathscr{R}(\mathbb{T},\mathbb{R})$, $c_{\bar m}\triangleleft c\triangleleft b-\sigma$, and $p\in\mathscr{P}$, the mapping $(\nu^{\bar m-1}_\tau,\upsilon^{\bar m-1}_\tau)(\cdot,p):\mathscr{X}\to\mathscr{B}^{\bar m}_{\tau,c}$ is differentiable with derivative*

$$D_1\begin{pmatrix}\nu^{\bar m-1}_\tau\\ \upsilon^{\bar m-1}_\tau\end{pmatrix}=\begin{pmatrix}\nu^{\bar m}_\tau\\ \upsilon^{\bar m}_\tau\end{pmatrix}:\mathscr{X}\times\mathscr{P}\longrightarrow\mathscr{B}^{\bar m}_{\tau,c}. \tag{4.22}$$

Let $c\in\mathscr{C}^+_{\mathrm{rd}}\mathscr{R}(\mathbb{T},\mathbb{R})$, $c_{\bar m}\triangleleft c\triangleleft b-\sigma$, and $p\in\mathscr{P}$ be fixed. First, we show that $(\nu^{\bar m-1}_\tau,\upsilon^{\bar m-1}_\tau)(\cdot,p)$ is differentiable and then we prove that the derivative is given by $(\nu^{\bar m}_\tau,\upsilon^{\bar m}_\tau)(\cdot,p):\mathscr{X}\to\mathscr{L}(\mathscr{X};\mathscr{B}^{\bar m-1}_{\tau,c})\cong\mathscr{B}^{\bar m}_{\tau,c}$ (cf. Lemma 2.3(c)). Thereto choose $\xi\in\mathscr{X}$ arbitrarily, but fixed. From now on, for the rest of the proof of Step IV, we suppress the p-dependence of the mappings under consideration; nevertheless $p\in\mathscr{P}$ is arbitrary. Using the fixed point equation (4.9) for $(\nu^{\bar m-1}_\tau,\upsilon^{\bar m-1}_\tau)$, we get for $h\in\mathscr{X}$ the identity

$$\begin{aligned}
&\begin{pmatrix}\nu^{\bar m-1}_\tau\\ \upsilon^{\bar m-1}_\tau\end{pmatrix}(t;\xi+h)-\begin{pmatrix}\nu^{\bar m-1}_\tau\\ \upsilon^{\bar m-1}_\tau\end{pmatrix}(t;\xi)\\
&=\begin{pmatrix}\displaystyle\int_\tau^t\Phi_A(t,\sigma(s))\left[D_{(2,3)}F(s,(\nu_\tau,\upsilon_\tau)(s;\xi+h))\begin{pmatrix}\nu^{\bar m-1}_\tau\\ \upsilon^{\bar m-1}_\tau\end{pmatrix}(s;\xi+h)+R^{\bar m-1}_1(s,\xi+h)\right]\Delta s\\
\displaystyle -\int_t^\infty\Phi_B(t,\sigma(s))\left[D_{(2,3)}G(s,(\nu_\tau,\upsilon_\tau)(s;\xi+h))\begin{pmatrix}\nu^{\bar m-1}_\tau\\ \upsilon^{\bar m-1}_\tau\end{pmatrix}(s;\xi+h)+R^{\bar m-1}_2(s,\xi+h)\right]\Delta s\end{pmatrix}\\
&\quad-\begin{pmatrix}\displaystyle\int_\tau^t\Phi_A(t,\sigma(s))\left[D_{(2,3)}F(s,(\nu_\tau,\upsilon_\tau)(s;\xi))\begin{pmatrix}\nu^{\bar m-1}_\tau\\ \upsilon^{\bar m-1}_\tau\end{pmatrix}(s;\xi)+R^{\bar m-1}_1(s,\xi)\right]\Delta s\\
\displaystyle -\int_t^\infty\Phi_B(t,\sigma(s))\left[D_{(2,3)}G(s,(\nu_\tau,\upsilon_\tau)(s;\xi))\begin{pmatrix}\nu^{\bar m-1}_\tau\\ \upsilon^{\bar m-1}_\tau\end{pmatrix}(s;\xi)+R^{\bar m-1}_2(s,\xi)\right]\Delta s\end{pmatrix}
\end{aligned} \tag{4.23}$$

by (4.10) for $t \in \mathbb{T}_\tau^+$. This leads to

$$\begin{aligned}
&\begin{pmatrix} \nu_\tau^{\bar{m}-1} \\ \upsilon_\tau^{\bar{m}-1} \end{pmatrix}(t;\xi+h) - \begin{pmatrix} \nu_\tau^{\bar{m}-1} \\ \upsilon_\tau^{\bar{m}-1} \end{pmatrix}(t;\xi) \\
&\quad - \begin{pmatrix} \displaystyle\int_\tau^t \Phi_A(t,\sigma(s)) D_{(2,3)}F(s,(\nu_\tau,\upsilon_\tau)(s;\xi+h)) \left[\begin{pmatrix} \nu_\tau^{\bar{m}-1} \\ \upsilon_\tau^{\bar{m}-1} \end{pmatrix}(s;\xi+h) - \begin{pmatrix} \nu_\tau^{\bar{m}-1} \\ \upsilon_\tau^{\bar{m}-1} \end{pmatrix}(s;\xi)\right]\Delta s \\
\displaystyle -\int_t^\infty \Phi_B(t,\sigma(s)) D_{(2,3)}G(s,(\nu_\tau,\upsilon_\tau)(s;\xi+h)) \left[\begin{pmatrix} \nu_\tau^{\bar{m}-1} \\ \upsilon_\tau^{\bar{m}-1} \end{pmatrix}(s;\xi+h) - \begin{pmatrix} \nu_\tau^{\bar{m}-1} \\ \upsilon_\tau^{\bar{m}-1} \end{pmatrix}(s;\xi)\right]\Delta s \end{pmatrix} \\
&= \begin{pmatrix} \displaystyle\int_\tau^t \Phi_A(t,\sigma(s)) \big[D_{(2,3)}F(s,(\nu_\tau,\upsilon_\tau)(s;\xi+h)) \\
\qquad\qquad -D_{(2,3)}F(s,(\nu_\tau,\upsilon_\tau)(s;\xi))\big] \begin{pmatrix} \nu_\tau^{\bar{m}-1} \\ \upsilon_\tau^{\bar{m}-1} \end{pmatrix}(s;\xi+h)\Delta s \\
\displaystyle -\int_t^\infty \Phi_B(t,\sigma(s)) \big[D_{(2,3)}G(s,(\nu_\tau,\upsilon_\tau)(s;\xi+h)) \\
\qquad\qquad -D_{(2,3)}G(s,(\nu_\tau,\upsilon_\tau)(s;\xi))\big] \begin{pmatrix} \nu_\tau^{\bar{m}-1} \\ \upsilon_\tau^{\bar{m}-1} \end{pmatrix}(s;\xi+h)\Delta s \end{pmatrix} \\
&\quad + \begin{pmatrix} \displaystyle\int_\tau^t \Phi_A(t,\sigma(s)) \big[R_1^{\bar{m}-1}(s,\xi+h) - R_1^{\bar{m}-1}(s,\xi)\big]\Delta s \\
\displaystyle -\int_t^\infty \Phi_B(t,\sigma(s)) \big[R_2^{\bar{m}-1}(s,\xi+h) - R_2^{\bar{m}-1}(s,\xi)\big]\Delta s \end{pmatrix} \quad \text{for } t \in \mathbb{T}_\tau^+.
\end{aligned} \tag{4.24}$$

With functions $(\nu^{\bar{m}-1}, \upsilon^{\bar{m}-1}) \in \mathcal{B}_{\tau,c}^{\bar{m}-1}$ and $h \in \mathcal{X}$, we define the operators

$$\mathcal{K} \in \mathcal{L}(\mathcal{B}_{\tau,c}^{\bar{m}-1}), \qquad \mathcal{E} \in \mathcal{L}(\mathcal{X};\mathcal{B}_{\tau,c}^{\bar{m}-1}), \qquad \mathcal{J} : \mathcal{X} \longrightarrow \mathcal{B}_{\tau,c}^{\bar{m}-1} \tag{4.25}$$

as follows:

$$\begin{aligned}
\mathcal{K}\begin{pmatrix} \nu^{\bar{m}-1} \\ \upsilon^{\bar{m}-1} \end{pmatrix} &:= \begin{pmatrix} \displaystyle\int_\tau^\cdot \Phi_A(\cdot,\sigma(s)) D_{(2,3)}F(s,(\nu_\tau,\upsilon_\tau)(s;\xi)) \begin{pmatrix} \nu^{\bar{m}-1} \\ \upsilon^{\bar{m}-1} \end{pmatrix}(s)\Delta s \\
\displaystyle -\int_\cdot^\infty \Phi_B(\cdot,\sigma(s)) D_{(2,3)}G(s,(\nu_\tau,\upsilon_\tau)(s;\xi)) \begin{pmatrix} \nu^{\bar{m}-1} \\ \upsilon^{\bar{m}-1} \end{pmatrix}(s)\Delta s \end{pmatrix}, \\
\mathcal{E}h &:= \begin{pmatrix} \displaystyle\int_\tau^\cdot \Phi_A(\cdot,\sigma(s)) R_1^{\bar{m}}(s,\xi)\Delta s h \\
\displaystyle -\int_\cdot^\infty \Phi_B(\cdot,\sigma(s)) R_2^{\bar{m}}(s,\xi)\Delta s h \end{pmatrix},
\end{aligned} \tag{4.26}$$

$$\mathcal{J}(h) := \begin{pmatrix} \displaystyle\int_\tau^{\cdot} \Phi_A(\cdot,\sigma(s))\Big\{\big[D_{(2,3)}F(s,(\nu_\tau,\upsilon_\tau)(s;\xi+h)) - D_{(2,3)}F(s,(\nu_\tau,\upsilon_\tau)(s;\xi))\big] \\ \qquad\qquad \cdot \begin{pmatrix} \nu_\tau^{\bar{m}-1} \\ \upsilon_\tau^{\bar{m}-1} \end{pmatrix}(s;\xi+h) + R_1^{\bar{m}-1}(s,\xi+h) - R_1^{\bar{m}-1}(s,\xi) - R_1^{\bar{m}}(s,\xi)h\Big\}\Delta s \\ -\displaystyle\int_{\cdot}^{\infty} \Phi_B(\cdot,\sigma(s))\Big\{\big[D_{(2,3)}G(s,(\nu_\tau,\upsilon_\tau)(s;\xi+h)) - D_{(2,3)}G(s,(\nu_\tau,\upsilon_\tau)(s;\xi))\big] \\ \qquad\qquad \cdot \begin{pmatrix} \nu_\tau^{\bar{m}-1} \\ \upsilon_\tau^{\bar{m}-1} \end{pmatrix}(s;\xi+h) + R_2^{\bar{m}-1}(s,\xi+h) - R_2^{\bar{m}-1}(s,\xi) - R_2^{\bar{m}}(s,\xi)h\Big\}\Delta s \end{pmatrix}. \tag{4.27}$$

In the subsequent lines we will show that $\mathcal{K}$, $\mathcal{E}$, and $\mathcal{J}$ are well defined. Using (3.2) and (3.4), it is easy to see that $\mathcal{K} : \mathcal{B}_{\tau,c}^{\bar{m}-1} \to \mathcal{B}_{\tau,c}^{\bar{m}-1}$ is linear and satisfies the estimate

$$\left\| \mathcal{K}\begin{pmatrix} \nu^{\bar{m}-1} \\ \upsilon^{\bar{m}-1} \end{pmatrix} \right\|_{\tau,c}^{+} \le \max\left\{ \frac{K_1|F|_1}{\lfloor c-a \rfloor}, \frac{K_2|G|_1}{\lfloor b-c \rfloor} \right\} \left\| \begin{pmatrix} \nu^{\bar{m}-1} \\ \upsilon^{\bar{m}-1} \end{pmatrix} \right\|_{\tau,c}^{+} \le L \left\| \begin{pmatrix} \nu^{\bar{m}-1} \\ \upsilon^{\bar{m}-1} \end{pmatrix} \right\|_{\tau,c}^{+} \quad \text{by (3.24)} \tag{4.28}$$

which in turn gives us

$$\|\mathcal{K}\|_{\mathcal{L}(\mathcal{B}_{\tau,c}^{\bar{m}-1})} < 1 \quad \text{by (3.15)}. \tag{4.29}$$

Keeping in mind that $\mathcal{E}h = \mathcal{T}_\tau^{\bar{m}}(0;\xi,p)h$ (cf. (4.10)), Step II yields the inclusion $\mathcal{E}h \in \mathcal{B}_{\tau,c}^{\bar{m}-1}$, while $\mathcal{E}$ is obviously linear and continuous, hence $\mathcal{E} \in \mathcal{L}(\mathcal{X};\mathcal{B}_{\tau,c}^{\bar{m}-1})$. Arguments similar to those in Step II, together with (4.18), lead to $\mathcal{J}(h) \in \mathcal{B}_{\tau,c}^{\bar{m}-1}$ for any $h \in \mathcal{X}$. Because of (4.24), we obtain

$$\begin{aligned} &\left[\begin{pmatrix} \nu_\tau^{\bar{m}-1} \\ \upsilon_\tau^{\bar{m}-1} \end{pmatrix}(\xi+h) - \begin{pmatrix} \nu_\tau^{\bar{m}-1} \\ \upsilon_\tau^{\bar{m}-1} \end{pmatrix}(\xi)\right] \\ &\qquad - \mathcal{K}\left[\begin{pmatrix} \nu_\tau^{\bar{m}-1} \\ \upsilon_\tau^{\bar{m}-1} \end{pmatrix}(\xi+h) - \begin{pmatrix} \nu_\tau^{\bar{m}-1} \\ \upsilon_\tau^{\bar{m}-1} \end{pmatrix}(\xi)\right] = \mathcal{E}h + \mathcal{J}(h) \quad \text{for } h \in \mathcal{X}. \end{aligned} \tag{4.30}$$

Using the Neumann series (cf., e.g., [14, Theorem 2.1, page 74]) and the estimate (4.29), the linear mapping $I_{\mathcal{B}_{\tau,c}^{\bar{m}-1}} - \mathcal{K} \in \mathcal{L}(\mathcal{B}_{\tau,c}^{\bar{m}-1})$ is invertible and this implies

$$\begin{pmatrix} \nu_\tau^{\bar{m}-1} \\ \upsilon_\tau^{\bar{m}-1} \end{pmatrix}(\xi+h) - \begin{pmatrix} \nu_\tau^{\bar{m}-1} \\ \upsilon_\tau^{\bar{m}-1} \end{pmatrix}(\xi) = \left[I_{\mathcal{B}_{\tau,c}^{\bar{m}-1}} - \mathcal{K}\right]^{-1}\left[\mathcal{E}h + \mathcal{J}(h)\right] \quad \text{for } h \in \mathcal{X}. \tag{4.31}$$

Consequently, it remains to show $\lim_{h\to 0}(\mathcal{J}(h)/\|h\|) = 0$ in $\mathcal{B}_{\tau,c}^{\bar{m}-1}$, because then one gets

$$\lim_{h\to 0} \frac{1}{\|h\|} \left\| \begin{pmatrix} \nu_\tau^{\bar{m}-1} \\ \upsilon_\tau^{\bar{m}-1} \end{pmatrix}(\xi+h) - \begin{pmatrix} \nu_\tau^{\bar{m}-1} \\ \upsilon_\tau^{\bar{m}-1} \end{pmatrix}(\xi) - \left[I_{\mathcal{B}_{\tau,c}^{\bar{m}-1}} - \mathcal{K}\right]^{-1}\mathcal{E}h \right\|_{\tau,c}^{+} = 0, \tag{4.32}$$

that is, the claim of Step IV follows. Nevertheless the proof of $\lim_{h\to 0}(\|\mathcal{J}(h)\|^+_{\tau,c}/\|h\|)=0$ needs a certain technical effort. Thereto we use the fact that due to the induction hypothesis $A(\bar{m}-1)(d)$, the remainder

$$R^{\bar{m}-1}(s,\xi)=\sum_{j=1}^{\bar{m}-2}\binom{\bar{m}-2}{j}\frac{\partial^j}{\partial\xi^j}\left[D_{(2,3)}(F,G)(s,(\nu_\tau,\upsilon_\tau)(s;\xi))\right]\begin{pmatrix}\nu_\tau^{\bar{m}-1-j}\\ \upsilon_\tau^{\bar{m}-1-j}\end{pmatrix}(s;\xi)\quad\text{by (4.11)}\tag{4.33}$$

is partially differentiable with respect to $\xi\in\mathscr{X}$, where the derivative is given by

$$D_2R^{\bar{m}-1}(s,\xi)=R^{\bar{m}}(s,\xi)-D^2_{(2,3)}(F,G)(s,(\nu_\tau,\upsilon_\tau)(s;\xi))\begin{pmatrix}\nu_\tau^1\\ \upsilon_\tau^1\end{pmatrix}(s;\xi)\begin{pmatrix}\nu_\tau^{\bar{m}-1}\\ \upsilon_\tau^{\bar{m}-1}\end{pmatrix}(s;\xi)\quad\text{by (4.11).}\tag{4.34}$$

Using the abbreviation

$$\begin{aligned}\Delta R^{\bar{m}-1}&(s,\xi,h)\\ &:=\frac{1}{\|h\|}\Bigg\{R^{\bar{m}-1}(s,\xi+h)-R^{\bar{m}-1}(s,\xi)\\ &\qquad-\left[R^{\bar{m}}(s,\xi)-D^2_{(2,3)}(F,G)(s,(\nu_\tau,\upsilon_\tau)(s;\xi))\begin{pmatrix}\nu_\tau^1\\ \upsilon_\tau^1\end{pmatrix}(s;\xi)\begin{pmatrix}\nu_\tau^{\bar{m}-1}\\ \upsilon_\tau^{\bar{m}-1}\end{pmatrix}(s;\xi)\right]h\Bigg\},\end{aligned}\tag{4.35}$$

we obtain the limit relation $\lim_{h\to 0}\Delta R^{\bar{m}-1}(s,\xi,h)=0$ for $s\in\mathbb{T}^+_\tau$. Now we prove estimates for the components $\mathcal{J}_1$ and $\mathcal{J}_2$ of $\mathcal{J}=(\mathcal{J}_1,\mathcal{J}_2)$ separately. Here we get

$$\begin{aligned}\mathcal{J}_1(h)=\int_\tau^{\cdot}\Phi_A(\cdot,\sigma(s))\Bigg\{&\left[D_{(2,3)}F(s,(\nu_\tau,\upsilon_\tau)(s;\xi+h))\right.\\ &\left.-D_{(2,3)}F(s,(\nu_\tau,\upsilon_\tau)(s;\xi))\right]\begin{pmatrix}\nu_\tau^{\bar{m}-1}\\ \upsilon_\tau^{\bar{m}-1}\end{pmatrix}(s;\xi+h)\\ &-D^2_{(2,3)}F(s,(\nu_\tau,\upsilon_\tau)(s;\xi))\begin{pmatrix}\nu_\tau^1\\ \upsilon_\tau^1\end{pmatrix}(s;\xi)\begin{pmatrix}\nu_\tau^{\bar{m}-1}\\ \upsilon_\tau^{\bar{m}-1}\end{pmatrix}(s;\xi)h\\ &+\Delta R_1^{\bar{m}-1}(s,\xi,h)\|h\|\Bigg\}\Delta s\quad\text{by (4.27),}\end{aligned}\tag{4.36}$$

where subtraction and addition of the expression

$$D^2_{(2,3)}F(s,(\nu_\tau,\upsilon_\tau)(s;\xi))\left[\begin{pmatrix}\nu_\tau\\ \upsilon_\tau\end{pmatrix}(s;\xi+h)-\begin{pmatrix}\nu_\tau\\ \upsilon_\tau\end{pmatrix}(s;\xi)-\begin{pmatrix}\nu_\tau^1\\ \upsilon_\tau^1\end{pmatrix}(s;\xi)h\right]\begin{pmatrix}\nu_\tau^{\bar{m}-1}\\ \upsilon_\tau^{\bar{m}-1}\end{pmatrix}(s;\xi+h)\tag{4.37}$$

lead to

$$
\begin{aligned}
&\mathscr{J}_1(h)\\
&\quad= \int_\tau^{\cdot} \Phi_A(\cdot,\sigma(s))\Bigg\{\Bigg[D_{(2,3)}F(s,(\nu_\tau,\upsilon_\tau)(s;\xi+h))\\
&\qquad\qquad\qquad\qquad - D_{(2,3)}F(s,(\nu_\tau,\upsilon_\tau)(s;\xi)) - D^2_{(2,3)}F(s,(\nu_\tau,\upsilon_\tau)(s;\xi))\\
&\qquad\qquad\qquad\qquad \times\left(\begin{pmatrix}\nu_\tau\\ \upsilon_\tau\end{pmatrix}(s;\xi+h) - \begin{pmatrix}\nu_\tau\\ \upsilon_\tau\end{pmatrix}(s;\xi)\right)\Bigg]\begin{pmatrix}\nu_\tau^{\bar m-1}\\ \upsilon_\tau^{\bar m-1}\end{pmatrix}(s;\xi+h)\\
&\qquad\qquad\qquad\qquad + D^2_{(2,3)}F(s,(\nu_\tau,\upsilon_\tau)(s;\xi))\\
&\qquad\qquad\qquad\qquad \times\left[\begin{pmatrix}\nu_\tau\\ \upsilon_\tau\end{pmatrix}(s;\xi+h) - \begin{pmatrix}\nu_\tau\\ \upsilon_\tau\end{pmatrix}(s;\xi) - \begin{pmatrix}\nu_\tau^1\\ \upsilon_\tau^1\end{pmatrix}(s;\xi)h\right]\begin{pmatrix}\nu_\tau^{\bar m-1}\\ \upsilon_\tau^{\bar m-1}\end{pmatrix}(s;\xi+h)\\
&\qquad\qquad\qquad\qquad + D^2_{(2,3)}F(s,(\nu_\tau,\upsilon_\tau)(s;\xi))\begin{pmatrix}\nu_\tau^1\\ \upsilon_\tau^1\end{pmatrix}(s;\xi)\\
&\qquad\qquad\qquad\qquad \times\left[\begin{pmatrix}\nu_\tau^{\bar m-1}\\ \upsilon_\tau^{\bar m-1}\end{pmatrix}(s;\xi+h) - \begin{pmatrix}\nu_\tau^{\bar m-1}\\ \upsilon_\tau^{\bar m-1}\end{pmatrix}(s;\xi)\right]h\\
&\qquad\qquad\qquad\qquad + \Delta R_1^{\bar m-1}(s,\xi,h)\|h\|\Bigg\}\Delta s \quad \text{for } t\in\mathbb{T}_\tau^+.
\end{aligned}
\tag{4.38}
$$

Using the quotient

$$
\begin{aligned}
&\Delta D_{(2,3)}F(s,x,y,h_1,h_2)\\
&\qquad := \frac{D_{(2,3)}F(s,x+h_1,y+h_2) - D_{(2,3)}F(s,x,y) - D^2_{(2,3)}F(s,x,y)\begin{pmatrix}h_1\\ h_2\end{pmatrix}}{\|(h_1,h_2)\|}
\end{aligned}
\tag{4.39}
$$

for $s\in\mathbb{T}$ and $x\in\mathscr{X}$, $y\in\mathscr{Y}$, $h_1\in\mathscr{X}\setminus\{0\}$, and $h_2\in\mathscr{Y}\setminus\{0\}$, we obtain the estimate

$$
\begin{aligned}
&\|(\mathscr{J}_1(h))(t)\|\\
&\le \int_\tau^t \|\Phi_A(t,\sigma(s))\|\Bigg[\|\Delta D_{(2,3)}F(s,(\nu_\tau,\upsilon_\tau)(s;\xi),(\nu_\tau,\upsilon_\tau)(s;\xi+h)-(\nu_\tau,\upsilon_\tau)(s;\xi))\|\\
&\qquad\qquad\qquad \cdot\left\|\begin{pmatrix}\nu_\tau\\ \upsilon_\tau\end{pmatrix}(s;\xi+h) - \begin{pmatrix}\nu_\tau\\ \upsilon_\tau\end{pmatrix}(s;\xi)\right\|\left\|\begin{pmatrix}\nu_\tau^{\bar m-1}\\ \upsilon_\tau^{\bar m-1}\end{pmatrix}(s;\xi+h)\right\|\\
&\qquad\qquad\qquad + \|D^2_{(2,3)}F(s,(\nu_\tau,\upsilon_\tau)(s;\xi))\|
\end{aligned}
$$

$$\cdot \left\| \begin{pmatrix} \nu_\tau \\ \upsilon_\tau \end{pmatrix}(s;\xi+h) - \begin{pmatrix} \nu_\tau \\ \upsilon_\tau \end{pmatrix}(s;\xi) - \begin{pmatrix} \nu_\tau^1 \\ \upsilon_\tau^1 \end{pmatrix}(s;\xi)h \right\| \left\| \begin{pmatrix} \nu_\tau^{\bar m-1} \\ \upsilon_\tau^{\bar m-1} \end{pmatrix}(s;\xi+h) \right\|$$

$$+ \| D^2_{(2,3)} F(s,(\nu_\tau,\upsilon_\tau)(s;\xi)) \| \left\| \begin{pmatrix} \nu_\tau^1 \\ \upsilon_\tau^1 \end{pmatrix}(s;\xi) \right\|$$

$$\cdot \left\| \left[\begin{pmatrix} \nu_\tau^{\bar m-1} \\ \upsilon_\tau^{\bar m-1} \end{pmatrix}(s;\xi+h) - \begin{pmatrix} \nu_\tau^{\bar m-1} \\ \upsilon_\tau^{\bar m-1} \end{pmatrix}(s;\xi) \right] h \right\|$$

$$+ \| \Delta R_1^{\bar m-1}(s;\xi,h) \| \|h\| \Bigg] \Delta s \quad \text{for } t \in \mathbb{T}_\tau^+. \tag{4.40}$$

With Hypothesis 3.1(ii) (cf. (3.2), (3.4)), the abbreviations (3.54), and the induction hypothesis $A(\bar m-1)(c)$, we therefore get

$$\begin{aligned} &\| (\mathcal{J}_1(h))(t) \| \\ &\le K_1 \int_\tau^t e_a(t,\sigma(s)) \Bigg[\| \Delta D_{(2,3)} F(s,(\nu_\tau,\upsilon_\tau)(s;\xi),(\nu_\tau,\upsilon_\tau)(s;\xi+h) - (\nu_\tau,\upsilon_\tau)(s;\xi)) \| \frac{1}{\|h\|} \\ &\qquad \cdot \left\| \begin{pmatrix} \nu_\tau \\ \upsilon_\tau \end{pmatrix}(s;\xi+h) - \begin{pmatrix} \nu_\tau \\ \upsilon_\tau \end{pmatrix}(s;\xi) \right\| C_{\bar m-1} e_{c_{\bar m-1}}(s,\tau) \\ &\qquad + |F|_2 \left\| \begin{pmatrix} \Delta\nu_\tau \\ \Delta\upsilon_\tau \end{pmatrix}(s,h) \right\| C_{\bar m-1} e_{c_{\bar m-1}}(s,\tau) \\ &\qquad + |F|_2 C_1 e_{c_1}(s,\tau) \left\| \begin{pmatrix} \nu_\tau^{\bar m-1} \\ \upsilon_\tau^{\bar m-1} \end{pmatrix}(s;\xi+h) - \begin{pmatrix} \nu_\tau^{\bar m-1} \\ \upsilon_\tau^{\bar m-1} \end{pmatrix}(s;\xi) \right\| \\ &\qquad + \| \Delta R_1^{\bar m-1}(s,\xi,h) \| \Bigg] \Delta s \|h\| \end{aligned} \tag{4.41}$$

for $t \in \mathbb{T}_\tau^+$. Rewriting this estimate and using Lemma 3.3, we obtain

$$\begin{aligned} \frac{\| \mathcal{J}_1(h) \|_{\tau,c}^+}{\|h\|} \le{} & K_1^2 C_{\bar m-1} \frac{\lfloor c-a \rfloor}{\lfloor c-a \rfloor - K_1 |F|_1} \sup_{\tau \preceq t} V_1(t,h) + K_1 |F|_2 C_{\bar m-1} \sup_{\tau \preceq t} V_2(t,h) \\ & + K_1 |F|_2 C_1 \sup_{\tau \preceq t} V_3(t,h) + K_1 \sup_{\tau \preceq t} V_4(t,h) \quad \text{by (3.6)} \end{aligned} \tag{4.42}$$

with

$$\begin{aligned} V_1(t,h) := e_c(\tau,t) \int_\tau^t & e_a(t,\sigma(s)) e_c(s,\tau) e_{\bar c_{m-1}}(s,\tau) \\ & \cdot \| \Delta D_{(2,3)} F(s,(\nu_\tau,\upsilon_\tau)(s;\xi),(\nu_\tau,\upsilon_\tau)(s;\xi+h) - (\nu_\tau,\upsilon_\tau)(s;\xi)) \| \Delta s, \end{aligned}$$

$$
\begin{aligned}
V_2(t,h) &:= e_c(\tau,t)\int_\tau^t e_a(t,\sigma(s))e_{\bar{c}_{m-1}}(s,\tau)\left\|\begin{pmatrix}\Delta\nu\\ \Delta\upsilon\end{pmatrix}(s,h)\right\|\Delta s,\\
V_3(t,h) &:= e_c(\tau,t)\int_\tau^t e_a(t,\sigma(s))e_{c_1}(s,\tau)\left\|\begin{pmatrix}\nu_\tau^{\bar{m}-1}\\ \upsilon_\tau^{\bar{m}-1}\end{pmatrix}(s;\xi+h)-\begin{pmatrix}\nu_\tau^{\bar{m}-1}\\ \upsilon_\tau^{\bar{m}-1}\end{pmatrix}(s;\xi)\right\|\Delta s,\\
V_4(t,h) &:= e_c(\tau,t)\int_\tau^t e_a(t,\sigma(s))\|\Delta R_1^{\bar{m}-1}(s,\xi,h)\|\Delta s.
\end{aligned}
\tag{4.43}
$$

Similar to Step 5 in the proof of Theorem 3.5, we get $\lim_{h\to 0}\sup_{\tau\preceq t}V_i(t,h)=0$ for $i\in\{1,\dots,4\}$, proving that $\lim_{h\to 0}(\|\mathcal{J}_1(h)\|_{\tau,c}^+/\|h\|)=0$. Completely analogously, one shows $\lim_{h\to 0}(\|\mathcal{J}_2(h)\|_{\tau,c}^+/\|h\|)=0$, and accordingly we have verified the differentiability of the mapping $(\nu_\tau^{\bar{m}-1},\upsilon_\tau^{\bar{m}-1})(\cdot,p):\mathscr{X}\to\mathscr{B}_{\tau,c}^{\bar{m}-1}$ for any $p\in\mathscr{P}$. Finally, we derive for any parameter $p\in\mathscr{P}$ that the derivative

$$
D_1\begin{pmatrix}\nu_\tau^{\bar{m}-1}\\ \upsilon_\tau^{\bar{m}-1}\end{pmatrix}(\cdot,p):\mathscr{X}\longrightarrow\mathscr{L}\left(\mathscr{X};\mathscr{B}_{\tau,c}^{\bar{m}-1}\right)\cong\mathscr{B}_{\tau,c}^{\bar{m}}
\tag{4.44}
$$

is the fixed point mapping $(\nu_\tau^{\bar{m}},\upsilon_\tau^{\bar{m}})(\cdot,p):\mathscr{X}\to\mathscr{B}_{\tau,c}^{\bar{m}}$ of $\mathscr{T}_\tau^{\bar{m}}(\cdot;\cdot,p)$. From the fixed point equation (4.9) for $(\nu_\tau^{\bar{m}-1},\upsilon_\tau^{\bar{m}-1})$, we obtain by partial differentiation with respect to $\xi\in\mathscr{X}$ the identity

$$
\begin{aligned}
&D_1\begin{pmatrix}\nu_\tau^{\bar{m}-1}\\ \upsilon_\tau^{\bar{m}-1}\end{pmatrix}(t;\xi,p)\\
&\quad=\begin{pmatrix}\displaystyle\int_\tau^t\Phi_A(t,\sigma(s))D_{(2,3)}F(s,(\nu_\tau,\upsilon_\tau)(s;\xi,p),p)D_1\begin{pmatrix}\nu_\tau^{\bar{m}-1}\\ \upsilon_\tau^{\bar{m}-1}\end{pmatrix}(s;\xi,p)\Delta s\\ \displaystyle -\int_t^\infty\Phi_A(t,\sigma(s))D_{(2,3)}G(s,(\nu_\tau,\upsilon_\tau)(s;\xi,p),p)D_1\begin{pmatrix}\nu_\tau^{\bar{m}-1}\\ \upsilon_\tau^{\bar{m}-1}\end{pmatrix}(s;\xi,p)\Delta s\end{pmatrix}\\
&\qquad+\begin{pmatrix}\displaystyle\int_\tau^t\Phi_A(t,\sigma(s))R_1^{\bar{m}}(s,\xi,p)\Delta s\\ \displaystyle -\int_t^\infty\Phi_B(t,\sigma(s))R_2^{\bar{m}}(s,\xi,p)\Delta s\end{pmatrix}\quad\text{for } t\in\mathbb{T}_\tau^+\quad\text{by (4.10).}
\end{aligned}
\tag{4.45}
$$

Hence the derivative $D_1(\nu_\tau^{\bar{m}-1},\upsilon_\tau^{\bar{m}-1})(\xi,p)\in\mathscr{L}(\mathscr{X};\mathscr{B}_{\tau,c}^{\bar{m}-1})\cong\mathscr{B}_{\tau,c}^{\bar{m}}$ (cf. Lemma 2.3(c)) is a fixed point of $\mathscr{T}_\tau^{\bar{m}}(\cdot;\xi,p)$ which in turn is unique by Step III, and consequently (4.22) holds.

Step V. Claim: *for every growth rate $c\in\mathscr{C}_{rd}^+\mathscr{R}(\mathbb{T},\mathbb{R})$, $c_{\bar{m}}\lhd c\lhd b-\sigma$, the mapping $D_1^{\bar{m}}(\nu_\tau,\upsilon_\tau):\mathscr{X}\times\mathscr{P}\to\mathscr{B}_{\tau,c}^{\bar{m}}$ is continuous, that is, $A(\bar{m})(d)$ holds.*

Because of (4.22), it suffices to prove the continuity of the mapping $(\nu_\tau^{\bar{m}},\upsilon_\tau^{\bar{m}}):\mathscr{X}\times\mathscr{P}\to\mathscr{B}_{\tau,c}^{\bar{m}}$. Let $c\in\mathscr{C}_{\mathrm{rd}}^+\mathscr{R}(\mathbb{T},\mathbb{R})$, $c_{\bar{m}}\lhd c\lhd b-\sigma$, and $\xi_0\in\mathscr{X}$, $p_0\in\mathscr{P}$ be arbitrary but fixed. From the fixed point equation (4.9) for $(\nu_\tau^{\bar{m}},\upsilon_\tau^{\bar{m}})$ and (3.2), (3.4), one gets for $\xi\in\mathscr{X}$ and $p\in\mathscr{P}$

the estimate

$$
\begin{aligned}
&\left\|\begin{pmatrix} \nu_\tau^{\bar m} \\ \upsilon_\tau^{\bar m}\end{pmatrix}(t;\xi,p)-\begin{pmatrix} \nu_\tau^{\bar m} \\ \upsilon_\tau^{\bar m}\end{pmatrix}(t;\xi_0,p_0)\right\| \\
&\quad \le \max\Bigg\{K_1\int_\tau^t e_a(t,\sigma(s)) \\
&\qquad\qquad \times \Bigg\|D_{(2,3)}F(s,(\nu_\tau,\upsilon_\tau)(s;\xi,p),p)\begin{pmatrix} \nu_\tau^{\bar m} \\ \upsilon_\tau^{\bar m}\end{pmatrix}(s;\xi,p)+R_1^{\bar m}(s,\xi,p) \\
&\qquad\qquad\quad -D_{(2,3)}F(s,(\nu_\tau,\upsilon_\tau)(s;\xi_0,p_0),p_0)\begin{pmatrix} \nu_\tau^{\bar m} \\ \upsilon_\tau^{\bar m}\end{pmatrix}(s;\xi_0,p_0)-R_1^{\bar m}(s,\xi_0,p_0)\Bigg\|\Delta s, \\
&\qquad\qquad K_2\int_t^\infty e_b(t,\sigma(s))\Bigg\|D_{(2,3)}G(s,(\nu_\tau,\upsilon_\tau)(s;\xi,p),p)\begin{pmatrix} \nu_\tau^{\bar m} \\ \upsilon_\tau^{\bar m}\end{pmatrix}(s;\xi,p)+R_2^{\bar m}(s,\xi,p) \\
&\qquad\qquad\qquad -D_{(2,3)}G(s,(\nu_\tau,\upsilon_\tau)(s;\xi_0,p_0),p_0)\begin{pmatrix} \nu_\tau^{\bar m} \\ \upsilon_\tau^{\bar m}\end{pmatrix}(s;\xi_0,p_0) \\
&\qquad\qquad\qquad -R_2^{\bar m}(s,\xi_0,p_0)\Bigg\|\Delta s\Bigg\} \quad \text{for } t\in\mathbb{T}_\tau^+ \quad \text{by (4.10).}
\end{aligned}
\tag{4.46}
$$

Addition and subtraction of the expressions

$$
\begin{aligned}
&D_{(2,3)}F(s,(\nu_\tau,\upsilon_\tau)(s;\xi,p),p)\begin{pmatrix} \nu_\tau^{\bar m} \\ \upsilon_\tau^{\bar m}\end{pmatrix}(s;\xi_0,p_0), \\
&D_{(2,3)}G(s,(\nu_\tau,\upsilon_\tau)(s;\xi,p),p)\begin{pmatrix} \nu_\tau^{\bar m} \\ \upsilon_\tau^{\bar m}\end{pmatrix}(s;\xi_0,p_0),
\end{aligned}
\tag{4.47}
$$

respectively, in the corresponding norms lead to

$$
\left\|\begin{pmatrix} \nu_\tau^{\bar m} \\ \upsilon_\tau^{\bar m}\end{pmatrix}(t;\xi,p)-\begin{pmatrix} \nu_\tau^{\bar m} \\ \upsilon_\tau^{\bar m}\end{pmatrix}(t;\xi_0,p_0)\right\| \le \max\{\alpha+\beta,\gamma+\delta\}
\tag{4.48}
$$

with the abbreviations

$$
\begin{aligned}
\alpha &:= K_1\int_\tau^t e_a(t,\sigma(s))\left[\|\hat F(s,\xi,p)\|\left\|\begin{pmatrix} \nu_\tau^{\bar m} \\ \upsilon_\tau^{\bar m}\end{pmatrix}(s;\xi_0,p_0)\right\|+\|R_1^{\bar m}(s,\xi,p)-R_1^{\bar m}(s,\xi_0,p_0)\|\right]\Delta s, \\
\beta &:= K_1|F|_1\int_\tau^t e_a(t,\sigma(s))\left\|\begin{pmatrix} \nu_\tau^{\bar m} \\ \upsilon_\tau^{\bar m}\end{pmatrix}(s;\xi,p)-\begin{pmatrix} \nu_\tau^{\bar m} \\ \upsilon_\tau^{\bar m}\end{pmatrix}(s;\xi_0,p_0)\right\|\Delta s, \\
\gamma &:= K_2\int_t^\infty e_b(t,\sigma(s))\left[\|\hat G(s,\xi,p)\|\left\|\begin{pmatrix} \nu_\tau^{\bar m} \\ \upsilon_\tau^{\bar m}\end{pmatrix}(s;\xi_0,p_0)\right\|+\|R_2^{\bar m}(s,\xi,p)-R_2^{\bar m}(s,\xi_0,p_0)\|\right]\Delta s, \\
\delta &:= K_2|G|_1\int_t^\infty e_b(t,\sigma(s))\left\|\begin{pmatrix} \nu_\tau^{\bar m} \\ \upsilon_\tau^{\bar m}\end{pmatrix}(s;\xi,p)-\begin{pmatrix} \nu_\tau^{\bar m} \\ \upsilon_\tau^{\bar m}\end{pmatrix}(s;\xi_0,p_0)\right\|\Delta s,
\end{aligned}
\tag{4.49}
$$

and $\hat{F}$, $\hat{G}$ given by (3.79). Using again relation (3.29), we obtain

$$\begin{aligned}
&\left\|\begin{pmatrix} \nu_\tau^{\bar{m}} \\ \upsilon_\tau^{\bar{m}} \end{pmatrix}(t;\xi,p) - \begin{pmatrix} \nu_\tau^{\bar{m}} \\ \upsilon_\tau^{\bar{m}} \end{pmatrix}(t;\xi_0,p_0)\right\| e_c(\tau,t) \\
&\qquad \le \alpha e_c(\tau,t) + \gamma e_c(\tau,t) + L\left\|\begin{pmatrix} \nu_\tau^{\bar{m}} \\ \upsilon_\tau^{\bar{m}} \end{pmatrix}(\xi,p) - \begin{pmatrix} \nu_\tau^{\bar{m}} \\ \upsilon_\tau^{\bar{m}} \end{pmatrix}(\xi_0,p_0)\right\|_{\tau,c}^{+} \quad \text{by (3.24)}
\end{aligned} \tag{4.50}$$

for $t \in \mathbb{T}_\tau^+$. Passing over to the least upper bound over $t \in \mathbb{T}_\tau^+$ yields (cf. (3.15))

$$\left\|\begin{pmatrix} \nu_\tau^{\bar{m}} \\ \upsilon_\tau^{\bar{m}} \end{pmatrix}(\xi,p) - \begin{pmatrix} \nu_\tau^{\bar{m}} \\ \upsilon_\tau^{\bar{m}} \end{pmatrix}(\xi_0,p_0)\right\|_{\tau,c}^{+} \le \frac{\max\{K_1,K_2\}}{1-L} \sup_{\tau \preceq t} W(t,\xi,p) \tag{4.51}$$

with

$$\begin{aligned}
W(t,\xi,p) := {}& \int_\tau^t e_a(t,\sigma(s)) e_c(s,\tau)\Bigg[\|\hat{F}(s,\xi,p)\|\left\|\begin{pmatrix} \nu_\tau^{\bar{m}} \\ \upsilon_\tau^{\bar{m}} \end{pmatrix}(s;\xi_0,p_0)\right\| \\
&\qquad + \|R_1^{\bar{m}}(s,\xi,p) - R_1^{\bar{m}}(s,\xi_0,p_0)\|\Bigg]\Delta s \\
&+ \int_t^\infty e_b(t,\sigma(s)) e_c(s,\tau)\Bigg[\|\hat{G}(s,\xi,p)\|\left\|\begin{pmatrix} \nu_\tau^{\bar{m}} \\ \upsilon_\tau^{\bar{m}} \end{pmatrix}(s;\xi_0,p_0)\right\| \\
&\qquad + \|R_2^{\bar{m}}(s,\xi,p) - R_2^{\bar{m}}(s,\xi_0,p_0)\|\Bigg]\Delta s.
\end{aligned} \tag{4.52}$$

Using the two limit relations

$$\lim_{(\xi,p)\to(\xi_0,p_0)} \left\|\begin{pmatrix} \hat{F} \\ \hat{G} \end{pmatrix}(s,\xi,p)\right\| = 0, \qquad \lim_{(\xi,p)\to(\xi_0,p_0)} \|R^{\bar{m}}(s,\xi,p) - R^{\bar{m}}(s,\xi_0,p_0)\| = 0, \quad \text{for } s \in \mathbb{T}_\tau^+, \tag{4.53}$$

where the first one follows by the continuity of $(\nu_\tau,\upsilon_\tau)(t;\cdot) : \mathscr{X}\times\mathscr{P} \to \mathscr{X}\times\mathscr{Y}$, $t \in \mathbb{T}_\tau^+$, (cf. Step 1 in the proof of Theorem 3.5) and $D_{(2,3)}(F,G)$, and the latter one by our induction hypothesis $A(\bar{m}-1)(d)$, we finally obtain, similar to the proof of (3.84), the desired $\lim_{(\xi,p)\to(\xi_0,p_0)} \sup_{\tau\preceq t} W(t,\xi,p) = 0$. This yields our claim in Step V, and summarizing, we have verified $A(\bar{m})$.

Step VI. In the preceding five steps we have shown that $(\nu_\tau,\upsilon_\tau) : \mathscr{X}\times\mathscr{P} \to \mathscr{B}_{\tau,c}^+(\mathscr{X}\times\mathscr{P})$ is m_s-times continuously partially differentiable with respect to its first argument. With the identity $s(\tau,\xi,p) = \upsilon_\tau(\xi,p)(\tau)$, the claim follows from properties of the evaluation map (see [18, Lemma 3.4]) and the global bound for the derivatives can be obtained using the fact

$$\|D_2^n s(\tau,\xi,p)\| = \|D_1^n \upsilon_\tau(\xi,p)(\tau)\| \le \|\upsilon_\tau^n(\xi,p)\|_{\tau,c}^{+} \le C_n \quad \text{for } \xi \in \mathscr{X},\ p \in \mathscr{P} \quad \text{by (4.20)}, \tag{4.54}$$

and $n \in \{1,\dots,m_s\}$. Hereby the expression for C_1 is a consequence of (3.51).

(b) The smoothness proof of the mapping $r : \mathbb{T} \times \mathcal{Y} \times \mathcal{P} \to \mathcal{X}$ is dual to the above considerations for s. A formal differentiation of the identity (3.90) with respect to $\eta \in \mathcal{Y}$ gives us a fixed point equation $(\nu_\tau^l, \upsilon_\tau^l)(\eta, p) = \bar{\mathcal{T}}_\tau^l((\nu_\tau^l, \upsilon_\tau^l)(\eta, p); \eta, p)$ with the right-hand side

$$\bar{\mathcal{T}}_\tau^l(\nu^l, \upsilon^l; \eta, p) := \begin{pmatrix} \displaystyle\int_{-\infty}^{\cdot} \Phi_A(\cdot, \sigma(s)) \left[D_{(2,3)} F(s, (\nu_\tau, \upsilon_\tau)(s; \eta, p), p) \begin{pmatrix} \nu^l \\ \upsilon^l \end{pmatrix} (s) + \bar{R}_1^l(s, \eta, p) \right] \Delta s \\ \displaystyle\int_{\tau}^{\cdot} \Phi_B(\cdot, \sigma(s)) \left[D_{(2,3)} G(s, (\nu_\tau, \upsilon_\tau)(s; \eta, p), p) \begin{pmatrix} \nu^l \\ \upsilon^l \end{pmatrix} (s) + \bar{R}_2^l(s, \eta, p) \right] \Delta s \end{pmatrix} \tag{4.55}$$

for $t \in \mathbb{T}_\tau^-$ and parameters $p \in \mathcal{P}$, where the remainder $\bar{R}^l = (\bar{R}_1^l, \bar{R}_2^l)$ allows representations analogous to (4.11) and (4.12). We omit the further details.

(c) The recursion for the global bounds $C_n \geq 0$, $n \in \{2, \ldots, m\}$, of $\|D_2^n s(\tau, \xi, p)\|$ in (4.8) is an obvious consequence of the estimate (4.19) from Step II of part (a) in the present proof. A dual argument shows that the solution of the fixed point equation for (4.55) is globally bounded by C_n as well, and an estimate analogous to (4.54) gives us the global bounds for the partial derivatives of r. Hence, we have shown assertion (c) and the proof of Theorem 4.2 is finished. □

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References

[1] B. Aulbach, *The fundamental existence theorem on invariant fiber bundles*, J. Differ. Equations Appl. **3** (1998), no. 5-6, 501–537.
[2] B. Aulbach, C. Pötzsche, and S. Siegmund, *A smoothness theorem for invariant fiber bundles*, J. Dynam. Differential Equations **14** (2002), no. 3, 519–547.
[3] B. Aulbach and T. Wanner, *Integral manifolds for Carathéodory type differential equations in Banach spaces*, Six Lectures on Dynamical Systems (Augsburg, 1994) (B. Aulbach and F. Colonius, eds.), World Scientific, New Jersey, 1996, pp. 45–119.
[4] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales. An Introduction with Applications*, Birkhäuser Boston, Massachusetts, 2001.
[5] N. Castañeda and R. Rosa, *Optimal estimates for the uncoupling of differential equations*, J. Dynam. Differential Equations **8** (1996), no. 1, 103–139.
[6] S.-N. Chow and K. Lu, *C^k centre unstable manifolds*, Proc. Roy. Soc. Edinburgh Sect. A **108** (1988), no. 3-4, 303–320.
[7] M. S. ElBialy, *On pseudo-stable and pseudo-unstable manifolds for maps*, J. Math. Anal. Appl. **232** (1999), no. 2, 229–258.
[8] S. Hilger, *Analysis on measure chains—a unified approach to continuous and discrete calculus*, Results Math. **18** (1990), no. 1-2, 18–56.
[9] ———, *Smoothness of invariant manifolds*, J. Funct. Anal. **106** (1992), no. 1, 95–129.

[10] ———, *Generalized theorem of Hartman-Grobman on measure chains*, J. Austral. Math. Soc. Ser. A **60** (1996), no. 2, 157–191.

[11] M. W. Hirsch, C. C. Pugh, and M. Shub, *Invariant Manifolds*, Lecture Notes in Mathematics, vol. 583, Springer-Verlag, Berlin, 1977.

[12] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Encyclopedia of Mathematics and Its Applications, vol. 54, Cambridge University Press, Cambridge, 1995.

[13] S. Keller, *Asymptotisches Verhalten invarianter Faserbündel bei Diskretisierung und Mittelwertbildung im Rahmen der Analysis auf Zeitskalen*, Ph.D. thesis, Universität Augsburg, Augsburg, 1999.

[14] S. Lang, *Real and Functional Analysis*, 3rd ed., Graduate Texts in Mathematics, vol. 142, Springer-Verlag, New York, 1993.

[15] J. R. Munkres, *Topology: a First Course*, Prentice-Hall, New Jersey, 1975.

[16] L. Neidhart, *Integration im Rahmen des Maßkettenkalküls*, Diploma thesis, Universität Augsburg, Augsburg, 2001.

[17] C. Pötzsche, *Langsame Faserbündel dynamischer Gleichungen auf Maßketten*, Ph.D. thesis, Universität Augsburg, Augsburg, 2002.

[18] ———, *Pseudo-stable and pseudo-unstable fiber bundles for dynamic equations on measure chains*, J. Difference Equ. Appl. **9** (2003), no. 10, 947–968.

[19] ———, *Two perturbation results for semi-linear dynamic equations on measure chains*, New Progress in Difference Equations (B. Aulbach, S. Elaydi, and G. Ladas, eds.), CRC Press, Boca Raton, 2004.

[20] C. Pötzsche and S. Siegmund, *$\mathscr{C}^m$-smoothness of invariant fiber bundles*, to appear in Topol. Methods Nonlinear Anal.

[21] K. P. Rybakowski, *Formulas for higher-order Fréchet derivatives of composite maps, implicitly defined maps and solutions of differential equations*, Nonlinear Anal. **16** (1991), no. 6, 517–532.

[22] ———, *An abstract approach to smoothness of invariant manifolds*, Appl. Anal. **49** (1993), no. 1-2, 119–150.

[23] M. Shub, *Global Stability of Dynamical Systems*, Springer-Verlag, New York, 1987.

[24] S. Siegmund, *Spektraltheorie, glatte Faserungen und Normalformen für Differentialgleichungen vom Carathéodory-Typ*, Ph.D. thesis, Universität Augsburg, Augsburg, 1999.

[25] A. Vanderbauwhede, *Centre manifolds, normal forms and elementary bifurcations*, Dynamics Reported, Vol. 2, Dynam. Report. Ser. Dynam. Systems Appl., vol. 2, Wiley, Chichester, 1989, pp. 89–169.

[26] A. Vanderbauwhede and S. A. van Gils, *Center manifolds and contractions on a scale of Banach spaces*, J. Funct. Anal. **72** (1987), no. 2, 209–224.

Christian Pötzsche: Department of Mathematics, University of Augsburg, 86135 Augsburg, Germany

E-mail address: poetzsche@math.uni-augsburg.de

Stefan Siegmund: Department of Mathematics, J. W. Goethe University, Robert-Mayer-Straße 10, 60325 Frankfurt, Germany

E-mail address: siegmund@math.uni-frankfurt.de